33-231 **Physical Analysis** **Fall 2003**

Problems: Set 5 (due Wednesday, October 1, 2003)

20. A simple pendulum is made using a small body with mass m and a rigid rod with length L and negligible mass. It swings in a vertical plane in a uniform gravitational field. For the numerical parts of this problem, use the values $L = 1.00$ m and $g = 9.8$ m/s^2.

    a) Starting with the general differential equation we derived in class, obtain general expressions (in terms of g and L) for the period, frequency, and angular frequency, in the small-displacement approximation. From these, obtain numerical values using the above numbers.

    b) Suppose the mass is given an initial velocity v_o at its stable equilibrium position ($x = 0$). Derive, without approximations, a relation between v_o and the maximum vertical height above the equilibrium position reached by the particle. From this, derive an expression for the minimum value of v_o needed for the mass to go all the way around (i.e., past the vertical). Also obtain a numerical value, using the above numbers.

    c) Using the above values of L and g, obtain and plot (x as a function of t) a numerical solution of the differential equation (without approximations) if the initial conditions are $x_o = 0$ and $v_o = 0.1$ m/s. From your graph, determine the *period* of the motion, and compare your result with part (a) above.

    d) Repeat (c) using larger values of v_o. By trial and error, find approximately the value of v_o for which the period is 10% greater than the approximate small-displacement value found in (a) and (c).

    e) Repeat (d), using a value of v_o somewhat greater than the value found in (b). You should find that the graph looks qualitatively different. Discuss what is happening.

    f) Using $x_o = 0$ and $v_o = 0.100$ m/s, obtain a *phase plot* for the motion. Repeat using $v_o = 6.0$ m/s. Comment on the shapes of the two plots.

21. A particle with mass m moves along the x axis under the action of a force given by

$$F = -k\left(\frac{x}{a}\right)^{19}, \text{ where } k \text{ and } a \text{ are positive constants.}$$

a) Without using Maple, sketch a graph of F as a function of x. Don't try to be precise, but do label any particularly noteworthy points on the axes.

b) Obtain an expression for the potential energy function $V(x)$.

c) For the remaining parts of this problem, assume $m = 1$, $a = 1$, and $k = 1$. What are the *units* (SI) of these three quantities? (It is reasonable to choose units for a such that (x/a) is dimensionless.)

d) Use Maple to plot graphs of $F(x)$ and $V(x)$. Choose your range of values of x so that the numbers on the vertical scale are in a reasonable range.

e) Set up the differential equation for the motion, and use Maple to obtain a numerical solution. Take the initial position to be $x_o = 0$, and use several different values of the initial velocity v_o. Plot a graph of x as a function of t for each, and note how the period T of the motion varies with v_o.

f) For one of the values of v_o, obtain a phase plot (v as a function of x). How does it differ from the phase plot for a harmonic oscillator?

g) Note that the motion resembles that of a ball bouncing back and forth between two rigid walls. How would the period T of that motion be related to v_o? Work out a general expression for T in terms of m, a, and v_o. Compare your result with the dependence of T on v_o that you found in part (e).

h) (optional) Use Maple to obtain a graph of the force F on the particle, as a function of t (not x).

22. a) A critically damped harmonic oscillator has initial position x_o and initial velocity v_o (at time $t = 0$). In terms of these initial conditions, derive a general expression for x as a function of t.

b) Derive the expression requested in part (a), for the case of an *overdamped* oscillator.

23. For an underdamped harmonic oscillator, one possible motion is described by

$$x(t) = Ae^{-\gamma t}\cos\omega_d t, \quad \text{where } \omega_d = \sqrt{\omega_o{}^2 - \gamma^2}.$$

Using this function, show that the ratio of two successive maxima in displacement is constant. Note that the maxima *do not* occur at the maxima of the function $\cos\omega_d t$.

Hints: To find the maxima, take the derivative of $x(t)$ and set it equal to zero. Hence show that successive maxima are separated by a time interval $2\pi/\omega_d$. Thus the factor $\cos\omega_d t$ has the same value at all maxima, so the ratio of the values of x at two successive maxima is determined by the ratio of values of the exponential function.

24. An underdamped harmonic oscillator has a frequency 12/13 as great as it would have if there were no damping.

a) Find γ (expressed as a multiple of ω_o).

b) Find the fractional decrease in amplitude during one cycle (i.e., the peak of one cycle minus the peak of the next cycle, divided by the peak of the first), in terms of ω_o. The analysis in Problem 23 may be useful.

c) Find the fractional decrease in total energy during one cycle, (i.e., the decrease in E during one cycle, divided by the value of E at the beginning of the cycle) in terms of ω_o.

d) Determine the value of Q for this system.