33-231 **Physical Analysis** **Fall 2003**

Problems: Set 2 (due Wednesday, September 10, 2003

6. Consider the differential equation $\frac{dx}{dt} = -ax^2 t.$

   a) If x is a distance (measured in meters) and t is time (in seconds), what are the units of the constant a?

   b) Obtain the general solution of this equation by separation of variables, without using Maple. Your result should contain an arbitrary constant.

   c) Obtain the general solution of this equation by using Maple. Also use Maple to substitute your result back into the equation to verify that it is indeed a solution.

   d) Suppose that $a = 4$ (with appropriate units) and that at time $t = 0$, x has the value 2 m. (I.e., $x(0) = 2$ m.) Evaluate the arbitrary constant in your result from (a), without using Maple.

   e) Using the same numbers as in (d), use Maple to solve the equation and the initial condition together. Compare your result with that in (d).

   f) Using the result of (d) or (e), plot a graph of x as a function of t, using Maple.

   g) Using the result of (d) or (e), find the value of t for which $x = 1$ m, using the Maple `fsolve` command.

7. We discussed in class (on Wednesday, September 3) the problem of a body moving vertically under the action of its (constant) weight mg and an air resistance force $-bv$ proportional to velocity.

   a) Use the method of separation of variables, derive the expressions quoted in class for $v(t)$ and $x(t)$, namely,

$$v = -\frac{mg}{b} + \left(v_o + \frac{mg}{b}\right)e^{-(b/m)t},$$

$$y = y_o - \frac{mg}{b}t + \frac{m}{b}\left(v_o + \frac{mg}{b}\right)\left(1 - e^{-(b/m)t}\right).$$

   b) Derive the results of part (a) using Maple.

   c) Using the notation introduced in class, namely, $v_T = \dfrac{mg}{b}$, $\tau = \dfrac{m}{b}$,

   derive the alternative forms

$$v = -v_T + (v_o + v_T)e^{-t/\tau},$$

$$y = y_o - v_T t + (v_o + v_T)\,\tau\,(1 - e^{-t/\tau}).$$

8. Consider again the situation of Problem 7.

   a) Work out the Taylor series expansion of the function $e^{-(b/m)t}$, about $t = 0$, keeping terms only up to order t^2.

   b) Substitute the series expansion of the exponential into the expression for $v(t)$ derived in Problem 7. Use the first version (with the constants m, g, and b), not the later version (in terms of v_T and τ). Show that the resulting expression for $v(t)$ reduces to the no-friction result when $b \to 0$.

   c) Make the same substitution into the expression for $y(t)$, and again show that the result reduces to the no-friction result when $b \to 0$.

   Note: For any finite value of t, the product bt goes to zero as $b \to 0$. Any terms containing b that don't cancel can be discarded in this limit.

9. This problem includes some applications of *hyperbolic functions*. If you aren't familiar with hyperbolic functions, I suggest you review their definitions and properties, e.g., in Stewart (4th ed.), pp. 246-251. Use Maple to plot graphs of $\sinh x$, $\cosh x$, $\tanh x$, and $\operatorname{arctanh} x$ (which Stewart calls $\tanh^{-1} x$), to get a little familiarity with the properties of these functions. Note that $\operatorname{arctanh} x$ is *not* multivalued (unlike $\arctan x$) and that its asymptotic value at large $|x|$ is 1 .

   a) A sky diver falls vertically downward in a straight line. He starts at the origin with zero initial velocity, and the air drag force is proportional to v^2, i.e., $F = -bv^2$. Take the positive direction to be *downward*. With this choice, write the differential equation for v (from Newton's second law). Be sure you get the signs right. Without solving this equation, derive an expression for the terminal velocity v_T in terms of m, g, and b. Determine the units of b, and check to see that your expression for v_T has the right units.

   b) Separate variables and integrate, using the stated initial conditions but without using Maple. You will need (possibly with a change of variable) the following integral:

$$\int \frac{dz}{a^2 - z^2} = \frac{1}{a} \operatorname{arctanh} \frac{z}{a} = \frac{1}{a} \ln \left| \frac{a+z}{a-z} \right|.$$

   (The two forms are equivalent, although that's not obvious. The first is the best one for our purposes.)

   Re-arrange the result to get an expression for v as a function of t. You should find that v is a constant times a hyperbolic tangent of the quantity (t/τ), where $\tau = \sqrt{m/gb}$ Verify that τ has units of *time*, as it must for unit consistency. This quantity plays the role of a decay constant, describing the time involved in the approach to terminal velocity.

   c) Separate variables again and integrate to get an expression for x as a function of t. Substitute your result back into the equation to verify that it is a solution. Express the functions $v(t)$ and $x(t)$ in two ways: In terms of the constants m, g, and b, and in terms of v_T and τ. You should obtain

$$v = v_T \tanh \frac{t}{\tau} \qquad \text{and} \qquad x = v_T \tau \ln\left[\cosh\left(\frac{t}{\tau}\right)\right].$$

   d) (optional) Solve the differential equation for v using Maple. Your result may look rather different from your result in (b). Use Maple to substitute it back into the equation to verify that it is a solution. Try to show that the two results are equivalent, but don't spend too much time on this; it's a little tricky.

(continued)

9. (continued)

e) As in Problem 8, we'd like to check whether these results for $v(t)$ and $x(t)$ reduce to the "no-drag" results (i.e., $x = gt^2/2$ and $v = gt$) when $b \to 0$. Again we find that if we simply substitute $b = 0$, the results are indeterminate. So instead we expand the hyperbolic functions in a Taylor series. This time we'll use the Maple command `taylor(expression, t = 0, n)`. Check the Maple help file for details (i.e., type `?taylor`). Note that `n` is the number of terms Maple computes. Keeping only terms up to and including t^2 (i.e., up to $n = 3$), show that we do indeed get the expected results in the limit $b \to 0$.

10. This is a continuation of Problem 9, with some more or less realistic numbers.

a) A simple result from fluid mechanics is that for turbulent flow of a fluid (such as air) past an obstacle at speed v, the drag force F is given approximately by $F = \frac{1}{2} DA\rho v^2$, where ρ is the density of air (about 1.2 kg/m^3 at ordinary temperatures and pressures), A is the silhouette area of the object, as seen looking in the direction of flow, and D is a dimensionless fudge factor called the *drag coefficient*. Thus our constant b is given approximately by $b = DAr/2$.

For well-streamlined cars D can be as small as 0.35, but for a sky diver it is probably more like 0.8 to 0.9. For a sphere, it is about 0.44.

A reasonable guess for the silhouette area of a sky-diver falling while stretched out face-down in a horizontal plane is 1 m^2.

Putting all these things together, show that a reasonable guess for the value of b is 0.5 kg/m. Again, be sure to verify that the units come out correctly.

Using these values, and assuming $m = 80$ kg, work out numerical values for v_T and τ. Express v_T in m/s and in mi/hr. Do your results seem reasonable?

b) Finally, we'd like to find the time at which the sky-diver reaches 90% of terminal velocity, and the total distance he falls during this time. Use `fsolve`, `subs`, and whatever else you need, to get these values. Also use Maple to plot graphs of x and v as functions of time, and check whether they seem to be consistent with your results.

c) Afterthought: Several solid spheres are made from the same material, but they have different radii. They are dropped in a viscous fluid that applies a resisting force proportional to v^2. How does the terminal velocity of a sphere depend on its radius R? I.e., is it proportional to R, or inversely proportional, or what?