33-231 Physical Analysis Fall 2003

Problems: Set 14 (due Monday, December 1, 2003)

55. A rope with length L, held stationary at both ends, is vibrating in normal mode n with amplitude A_n, with a wave function given by Eq. (30).

 a) Substitute this wave function into Eq. (45) and integrate to obtain an expression for the total energy of the system, in terms of A_n and properties of the system. Verify that the total energy is constant.

 b) Suppose there are two normal modes present, so the wave function is the sum of two terms like Eq. (30) but with different values of n. Show that the total energy is the sum of two terms, one identified with each of the normal modes. Each term has the form given by the result in (a).

 Note: The integrals are rather messy if you do them by hand. I strongly recommend using Maple. The problem can be done using eight lines of Maple code. One caution: Maple has to simplify expressions such as $\sin(n\pi)$. It can't do that unless you tell it at the start that n is an integer. The appropriate Maple command is `assume(n, integer);` Also, I suggest telling Maple at the start that `c = sqrt(F/mu);`

56. A stretched rope is attached to a stationary point at $x = 0$ and extends in the $+x$ direction. An incoming sinusoidal wave, Eq. (18), is reflected at $x = 0$, producing an inverted reflected wave, Eq. (19), and the superposition of the two waves forms a standing wave, Eq. (22). When A is redefined, as in the paragraph following Eq. (29), the total wave function is

$$y(x, t) = A \sin kx \sin \omega t.$$

 a) Use Eq. (47) to derive an expression for the rate of energy transfer P past an arbitrary point x on the rope. Simplify your result by using the sine double-angle identity: $\sin(2\alpha) = 2 \sin \alpha \cos \alpha$.

 b) Derive an expression for the *maximum* rate of energy transfer past a given point. At what points does this maximum rate occur?

 c) Show that at each node and each antinode, the rate of transfer of energy is always zero. Explain in words how this comes about.

(over)

57. For a photon, the energy E and momentum magnitude p are related to the wave properties λ, k $(= 2\pi/\lambda)$, f, and ω $(= 2\pi f)$ by

$$E = hf = \hbar\omega, \qquad p = \frac{h}{\lambda} = \hbar k. \qquad \left(\hbar = \frac{h}{2\pi}.\right)$$

Both h and $\hbar$ are called Planck's constant; present-day usage favors $\hbar$.
In 1923 Louis de Broglie advanced the hypothesis that electrons should also have wave properties, and that the energy and momentum of an electron should be related to its wave properties in the same way as for a photon. The relation of energy to momentum is different, however. Photons are massless and move always with the speed of light c, and $E = pc$. For an electron with mass m, moving at a speed v much *less* than c, the corresponding relations are

$$p = mv \qquad \text{and} \qquad E = \tfrac{1}{2}mv^2 = \frac{p^2}{2m}.$$

In 1925 Erwin Schrödinger proposed that de Broglie's electrons should be described (in one dimension) by a wave function $\Psi(x,t) = e^{i(kx-\omega t)}$. He also proposed that this function should satisfy a wave equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = i\hbar\frac{\partial\Psi}{\partial t}.$$

a) Show that the wave function satisfies this equation if k and ω are related in a particular way, and that this relation of ω to k is consistent with the classical energy-momentum relation if $p = \hbar k$ and $E = \hbar\omega$.

b) Determine the phase velocity, $v = \omega/k$, of the de Broglie wave, and compare it with the classical velocity (p/m) of the particle. You should find a discrepancy of a factor of two.

c) Suppose several waves with slightly differing values of ω are superposed to form a wave pulse. The envelope of the pulse moves with a speed equal to the *group* velocity, $v_g = d\omega/dk$. Show that this is equal to the classical velocity of the particle.