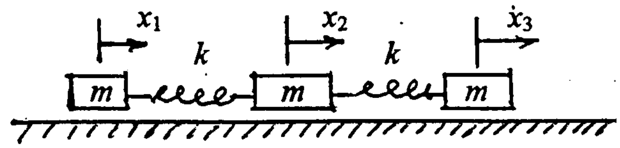
33-231 **Physical Analysis** **Fall 2003**

Problems: Set 10 (due Wednesday, November 5, 2003)

Note: In this problem set, several intermediate results are given with the problem statements, in order to help with error-hunting. Otherwise, if you make even a trivial error at the beginning, the whole problem will be derailed.

39. The system shown is meant to be a simple model of a linear triatomic molecule. For simplicity we make all three masses equal and let them move only along a straight line.



a) Write the $\Sigma F = ma$ equations, using the coordinates shown. Can you *guess* what the normal modes might look like, and what their frequencies might be?

b) From these equations, find the frequencies of the normal modes and the amplitude relation for each normal mode. In working out the secular determinant, don't be in too big a hurry to multiply out all the terms in full; first look for common factors such as $(k - \lambda m)$ (where $\lambda = m^2$).

$$\begin{vmatrix} k-\lambda m & -k & 0 \\ -k & 2k-\lambda m & -k \\ 0 & -k & k-\lambda m \end{vmatrix} = 0, \qquad \lambda_1 = 0, \qquad \lambda_2 = \frac{k}{m}, \qquad \lambda_3 = \frac{3k}{m}.$$

$$\lambda_1: \quad a_1 = a_2 = a_3.$$
$$\lambda_2: \quad a_1 = -a_3, \quad a_2 = 0.$$
$$\lambda_3: \quad a_1 = a_3, \quad a_2 = -2a_1.$$

c) Obtain a normal-coordinate transformation. Express the $\Sigma F = ma$ equations in terms of the normal coordinates, and show that they can be separated into three uncoupled equations, each of which contains only one normal coordinate. (You will have to multiply each equation by an appropriate numerical constant, and then add or subtract the equtions, as in Problem 38(d).) Solve the equations for the normal coordinates, and verify that you get the same normal-mode frequencies as in part (b).

$$x_1 = q_1 + q_2 + q_3,$$
$$x_2 = q_1 - 2q_3,$$
$$x_3 = q_1 - q_2 + q_3.$$

Substitute into equations of motion.

39 c) (continued)

Adding the three equations gives $3m\ddot{q}_1 = 0,\qquad \lambda_1 = \omega_1^2 = 0.$

The first minus the third gives: $-2kq_2 = 2m\ddot{q}_2,\qquad \lambda_2 = \omega_2^2 = \frac{k}{m}.$

First plus third minus twice second: $-18kq_3 = 6m\ddot{q}_3,\qquad \lambda_3 = \omega_3^2 = \frac{3k}{m}.$

d) Write an expression for the total energy of the system in terms of $x_1,\ x_2,\ x_3,$ and their derivatives. Then express this quantity in terms of the normal coordinates, and show that it can be separated into three parts, each of which contains only one normal coordinate and its derivative. (The algebra is a little tedious; use Maple if you like.)

$$E = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) + \tfrac{1}{2}k(x_2 - x_1)^2 + \tfrac{1}{2}k(x_3 - x_2)^2.$$

$$E = \tfrac{3}{2}m\dot{q}_1^2 + (m\dot{q}_2^2 + kq_2^2) + (3m\dot{q}_3^2 + 9kq_3^2).$$

e) Express the normal-coordinate transformation in terms of a matrix equation, as in Problem 38(f).

e) $\mathbf{x} = \mathbf{A}\mathbf{q}$: $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix},\qquad \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}.$$

$\mathbf{q} = \mathbf{A}^{-1}\mathbf{x}$: $$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\qquad \mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}.$$

Check: $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$

40. We want to explore the possibility of forced oscillations for the two-mass, three-spring system discussed extensively in class and in the "Coupled Oscillators" notes. Suppose that an additional force $F = F_o \cos \omega t$ is applied to the mass with coordinate x_1 (where F_o and ω are given, and ω is *not* one of the normal-mode frequencies).

a) Set up the equations of motion, using the coordinates used in class. Make sure you have the signs right.

b) Try a solution in the form $x_1 = a_1 \cos \omega t, \quad x_2 = a_2 \cos \omega t,$ where ω is the frequency of the driving force. Obtain expressions for a_1 and a_2, which will be functions of ω.

Equations are the same as Eqs. (1) in "*Coupled Oscillators and Normal Modes*" except for the added term $F_o \cos \omega t$ in the first equation.

Substitute $x_1 = a_1 \cos \omega t, \quad x_2 = a_2 \cos \omega t;$ collect terms and simplify.

Note that the resulting equations are *not* homogeneous, and the determinant of the system is *not* zero. Now a_1 and a_2 are multiples of F_o. Solve using determinants:

$$a_1 = \frac{\left(k + k' - m\omega^2\right)F_o}{\left(k + k' - m\omega^2\right)^2 - k'^2}, \qquad a_2 = \frac{k' F_o}{\left(k + k' - m\omega^2\right)^2 - k'^2}.$$

c) Assume some simple numerical values for the parameters, and plot a graph of each a as a function of frequency, for a range of frequencies that includes both normal-mode frequencies. Comment on your results.

Simple numerical example: Take $F_o = m = k = k' = 1$. Use Maple to plot a_1 and a_2 as functions of ω.

41. Again we consider the two-mass, three-spring system discussed in class and in the notes. We take away the driving force and install a shock absorber between the two masses (i.e., parallel to the spring k'). The effect of the shock absorber is to apply a force to each mass, with magnitude bv, where b is a constant and v is the *relative* speed of the two masses, that is, the speed with which the ends of the shock absorber are being pulled apart or pushed together.

a) Set up the equations of motion in terms of the same coordinates used in class.

Equations are the same as Eqs. (1) in "*Coupled Oscillators and Normal Modes*" except for the added term in each equation representing the damping force. This term has the form $\pm b(\dot{x}_2 - \dot{x}_1)$. Make sure you get the signs right.

b) Using the same normal-coordinate transformation used in class, express the equations of motion in terms of the normal coordinates.

Substitute normal-coordinate expressions: $x_1 = q_1 + q_2, \quad x_2 = q_1 - q_2.$

Add the two equations, and subtract the second from the first, to obtain

$$m\ddot{q}_1 + kq_1 = 0, \qquad m\ddot{q}_2 + 2b\dot{q}_2 + (k + 2k')q_2 = 0.$$

c) Solve the equations obtained in (b) to obtain the possible motions of this system. How many arbitrary constants does your solution contain? Comment on the significance of the *form* of your solutions. In particular, do they consist of two parts with somewhat different character? How might one choose initial conditions so that only one of the two parts is present?

The general solutions of the q equations are

$$q_1 = C\cos(\omega_1 t + \varphi_1), \quad q_2 = e^{-\gamma_2 t}\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right], \quad \text{where}$$

$$\gamma_2 = \frac{2b}{2m} = \frac{b}{m} \quad \text{and} \quad \omega_d = \sqrt{\omega_2{}^2 - \gamma_2{}^2}\,.$$

What do these results mean? How do they depend on initial conditions?

42. You are given the following two matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}.$$

For (a) and (b), do each calculation two ways: (1) by hand; (2) with Maple. Use Maple for (c) and (d)

a) Calculate the products **AB** and **BA.**

b) Calculate the sum **A + B.**

c) Obtain the inverse of **B**, i.e., $\mathbf{B}^{-1}$, and calculate $\mathbf{B}\,\mathbf{B}^{-1}$ and $\mathbf{B}^{-1}\mathbf{B}$.

d) Calculate the product $4\mathbf{B}^{-1}$.