11 Normal Modes -- Matrix Methods — Fall 2003

Matrix methods provide a very powerful tool for analyzing normal modes of coupled oscillator systems. We'll introduce this general approach by means of an example, the same system we discussed in detail in Section 8 of these notes. Our development will parallel the previous treatment, but we'll use matrix language.

Example

For the system discussed in Section 8, the equations of motion (from $\Sigma F = ma$) are

$$\begin{aligned} m\ddot{x}_1 &= -(k + k')x_1 + k'x_2, \\ m\ddot{x}_2 &= k'x_1 - (k + k')x_2. \end{aligned} \tag{1}$$

The equations of motion can be written as a single matrix equation:

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k + k' & -k' \\ -k' & k + k' \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2}$$

Each side of this equation is a column matrix. You should verify that when the matrix products are carried out, equating the first (top) elements on the two sides gives the first of Eqs. (1), and equating the second (bottom) elements gives the second.

We define the matrices **M**, **K**, and **x** as follows:

$$\mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} k + k' & -k' \\ -k' & k + k' \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{3}$$

Then Eq. (2) can be written simply as

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}. \tag{4}$$

For other systems with different arrangements of masses and springs, the **M** and **K** matrices will be different. But if the spring forces are *linear* functions of the coordinates, the equations of motion can always be written in the form of Eq. (4). So the following development is general, and is not restricted to the specific example cited above.

Note that the (1,1) element of **K** represents the negative of the force on mass 1 when it is displaced a distance x_1. The (1,2) element is the negative of the force on mass 1 when mass 2 is displaced a distance x_2, and so on. This gives an alternative way to determine the elements of **K**, instead of working out Eqs. (1) from $\Sigma F = ma$.

Caution: Some books define **K** with the opposite sign, so there is no minus in Eq. (4). (See, for example, Edwards and Penney, p. 321.) We prefer to retain the minus sign, so that Eq. (4) has the same form as the equation of motion $m\ddot{x} = -kx$ for a single particle.

General Formulation

For a normal mode, each x must vary sinusoidally, and all x's must have the same frequency. Therefore we try to find a solution of Eq. (4) in the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cos(\omega t + \varphi), \qquad \text{or} \qquad \mathbf{x} = C\mathbf{a}\cos(\omega t + \varphi). \tag{5}$$

In this expression, C is a scalar amplitude factor (determined by initial conditions), and $\mathbf{a}$ is a column matrix (or vector) whose elements a_1, a_2 give the ratios of the amplitudes of the x's for each normal mode. For example, if $a_2 = 2a_1$, the motion of the mass with coordinate x_2 is in phase with that of x_1, but with amplitude twice as great.

Taking the second time derivative of Eq. (5), we get

$$\ddot{\mathbf{x}} = -\omega^2 C\mathbf{a}\cos(\omega t + \varphi) = -\lambda C\mathbf{a}\cos(\omega t + \varphi), \qquad \text{where} \qquad \lambda = \omega^2. \tag{6}$$

To test whether Eq. (5) really *is* a solution of Eq. (4), we substitute Eqs. (5) and (6) into Eq. (4). After dividing out the common factor $C \cos(\omega t + \varphi)$, we get

$$-\lambda\mathbf{M}\mathbf{a} = -\mathbf{K}\mathbf{a}, \qquad \text{or} \qquad (\mathbf{K} - \lambda\mathbf{M})\mathbf{a} = 0. \tag{7}$$

This is a set of simultaneous, homogeneous equations for the elements of the column matrix $\mathbf{a}$. (This should sound familiar!) Non-trivial equations exist if, and only if, the determinant of the system is zero, that is, when

$$|\mathbf{K} - \lambda\mathbf{M}| = 0. \tag{8}$$

Thus Eq. (5) is a solution of Eq. (4) if, and only if, the value of $\lambda = \omega^2$ satisfies Eq. (8). This condition is called the *secular equation* for the system; it is satisfied only for certain particular values of λ, the roots of the secular equation.

For our example, we use the $\mathbf{M}$ and $\mathbf{K}$ matrices from Eqs. (3). The corresponding secular equation is:

$$\begin{vmatrix} k + k' - m\lambda & -k' \\ -k' & k + k' - m\lambda \end{vmatrix} = 0. \tag{9}$$

This agrees with Eq. (4) in Section 8 of these notes (page 8-2).

Equation (8) is an algebraic equation for λ. The degree of the equation (and hence the number of roots) is equal to the number of coordinates of the system (i.e., the number of *degrees of freedom*). In some problems two or more roots may be equal.

In our example, there are two roots:

$$\lambda_1 = \frac{k}{m}, \qquad \lambda_2 = \frac{k + 2k'}{m}. \tag{10}$$

The values of λ that satisfy Eq. (8) are called *eigenvalues*. There are as many values of λ as the number of degrees of freedom of the system. For each value of λ, there is a corresponding column matrix (or vector), giving the amplitude ratios for the x's for the corresponding normal mode. We'll call the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$; let $\mathbf{a}$ be the column matrix corresponding to λ_1, let $\mathbf{b}$ be the column matrix corresponding to λ_2, and so on. The column matrices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\cdots$ are the *eigenvectors* corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$, respectively. Then

$$(\mathbf{K} - \lambda_1\mathbf{M})\mathbf{a} = 0, \qquad (\mathbf{K} - \lambda_2\mathbf{M})\mathbf{b} = 0, \qquad (\mathbf{K} - \lambda_3\mathbf{M})\mathbf{c} = 0, \quad \cdots . \tag{11}$$

The most general solution of Eq. (4) can now be written as

$$\mathbf{x} = C_1\mathbf{a}\cos(\omega_1 t + \varphi_1) + C_2\mathbf{b}\cos(\omega_2 t + \varphi_2) + C_3\mathbf{c}\cos(\omega_3 t + \varphi_3) + \cdots. \tag{12}$$

The constants $C_1, C_2, C_3, \cdots$ and $\varphi_1, \varphi_2, \varphi_3 \cdots$ depend on the initial positions and velocities of the two masses), and the normal-mode frequencies are

$$\omega_1 = \sqrt{\lambda_1}, \qquad \omega_2 = \sqrt{\lambda_2}, \qquad \cdots .$$

As before, we define the normal coordinates for the system as

$$q_1 = C_1\cos(\omega_1 t + \varphi_1), \quad q_2 = C_2\cos(\omega_2 t + \varphi_2), \quad q_3 = C_3\cos(\omega_3 t + \varphi_3), \cdots . \tag{13}$$

Then the normal-coordinate transformation can be written as

$$\mathbf{x} = q_1\mathbf{a} + q_2\mathbf{b} + q_3\mathbf{c} + \cdots . \tag{14}$$

Now we define $\mathbf{q}$ to be a column matrix whose elements are the normal coordinates $q_1, q_2, q_3, \cdots$, and we define a square matrix $\mathbf{A}$ whose columns are the eigenvectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\cdots$. For two degrees of freedom,

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{15}$$

The normal coordinate transformation then becomes simply

$$\mathbf{x} = \mathbf{A}\,\mathbf{q} . \tag{16}$$

For our example,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

which agrees with the results at the end of Section 8 (page 8-5). (But note that there is a lot of arbitrariness in the definition of $\mathbf{A}$, because each q can be multiplied by an arbitrary numerical constant.)

This whole procedure is closely related to the discussion of eigenvalues and eigenvectors in Section 10 of these notes. Suppose we multiply both sides of Eq. (7) on the left by $\mathbf{M}^{-1}$. The result is

$$(\mathbf{M}^{-1}\mathbf{K} - \lambda\mathbf{M}^{-1}\mathbf{M})\mathbf{a} = 0, \qquad \text{or} \qquad (\mathbf{M}^{-1}\mathbf{K})\mathbf{a} = \lambda\mathbf{a}. \tag{17}$$

That is, $\mathbf{a}$ is an eigenvector of the matrix $(\mathbf{M}^{-1}\mathbf{K})$, with eigenvalue λ_1. Similarly, from Eq. (11), $\mathbf{b}$ is an eigenvector of $(\mathbf{M}^{-1}\mathbf{K})$ with eigenvalue λ_2, and so on.

If our objective is only to find the normal-mode frequencies and vibration patterns, we are finished. Each eigenvalue λ is the square of a normal-mode frequency ω, and each corresponding eigenvector gives the relative amplitudes of motion of the various masses for the normal-mode motion with this frequency.

However, there are problems in which we need to determine the constants C_1, C_2, C_3, $\cdots$ and $\varphi_1, \varphi_2, \varphi_3, \cdots$ in Eq. (12) in terms of the initial values of the coordinates and velocities. This would be fairly simple if the eigenvectors **a, b, c,** $\cdots$ were all mutually *orthogonal*. We could then multiply Eq. (12) successively by the transpose of each eigenvector, and all but one term on the right would be zero. We stated in Section 10 (page 10-4) that the eigenvectors of a *symmetric* matrix *are* all mutually orthogonal if they correspond to distinct eigenvalues.

In our example problem, the matrices $\mathbf{M}$, $\mathbf{M}^{-1}$, $\mathbf{K}$, and $\mathbf{M}^{-1}\mathbf{K}$ are all symmetric, and the eigenvectors *are* orthogonal. But this is an accident resulting from the symmetry of our physical system and our choice of coordinates. In general $\mathbf{M}$ and $\mathbf{K}$ are *not* symmetric; even when they *are* symmetric, the product $\mathbf{M}^{-1}\mathbf{K}$ in general is *not* symmetric. So in general the eigenvectors **a, b, c,** $\cdots$ are *not* mutually orthogonal. But it turns out that we can develop generalized concepts of normalization and orthogonality that are useful for fitting initial conditions and in other situations.

Normalization of Eigenvectors

We first generalize the concept of normalization. Because the eigenvectors $\mathbf{a}$ and $\mathbf{b}$ serve only to give the *ratios* of the amplitudes of motion of the various masses, for each normal mode, we are free to multiply each eigenvector by any constant. It turns out to be useful to multiply each eigenvector by a constant such that

$$\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{a} = 1, \qquad \mathbf{b}^{\mathrm{T}}\mathbf{M}\mathbf{b} = 1, \qquad \mathbf{c}^{\mathrm{T}}\mathbf{M}\mathbf{c} = 1, \qquad \text{and so on.} \tag{18}$$

(The more usual normalization condition is simply to require that $\mathbf{a}^{\mathrm{T}}\mathbf{a} = 1$, $\mathbf{b}^{\mathrm{T}}\mathbf{b} = 1$, and so on, or, $a_1^2 + a_2^2 + a_3^2 + \cdots = 1$, and so on.) When the eigenvectors are multiplied by factors such that they satisfy Eqs. (18), they are said to be *normalized,* in this generalized sense.

In our example, where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix},$$

the un-normalized $\mathbf{a}$ gives $\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{a} = 2m$. We multiply $\mathbf{a}$ by $1/\sqrt{2m}$; then $\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{a} = 1$. So in our example problem, the normalized eigenvector $\mathbf{a}$ corresponding to the eigenvalue $\lambda_1 = \sqrt{k/m}$ is

$$\mathbf{a} = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{19}$$

and the normalized eigenvector $\mathbf{b}$ corresponding to eigenvalue $\lambda_2 = \sqrt{(k + 2k')/m}$ is

$$\mathbf{b} = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{or} \qquad \mathbf{b} = \frac{1}{\sqrt{2m}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{20}$$

(Note that even with the normalization conditions, the eigenvectors are not determined uniquely; each one can always be multiplied by -1.)

Orthogonality of Eigenvectors

In the general case, if $\lambda_1, \lambda_2, \lambda_3, \cdots$ are all unequal, and if $\mathbf{M}$ and $\mathbf{K}$ are *symmetric* matrices (as in our example), then it can be shown that

$$\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{b} = 0, \quad \mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{c} = 0, \quad \mathbf{b}^{\mathrm{T}}\mathbf{M}\mathbf{c} = 0, \ \cdots . \tag{21}$$

Note that the *usual* definition of orthogonality of two vectors $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{a}^{\mathrm{T}}\mathbf{b} = 0$, i.e., the *scalar product* of $\mathbf{a}$ and $\mathbf{b}$ is zero. So Eq. (21) represents a generalization of the concepts of the scalar product and of orthogonality of two vectors. We invite you to verify that the eigenvectors $\mathbf{a}$ and $\mathbf{b}$ in our example problem do satisfy this condition.

If the system has three or more degrees of freedom, then any two eigenvectors that correspond to unequal eigenvalues are orthogonal in the sense of Eq. (21).

To prove Eqs. (21), we start with Eqs. (11):

$$\mathbf{K}\mathbf{a} = \lambda_1\mathbf{M}\mathbf{a} \qquad \text{and} \qquad \mathbf{K}\mathbf{b} = \lambda_2\mathbf{M}\mathbf{b}. \tag{22}$$

Multiply the first equation on the left by $\mathbf{b}^{\mathrm{T}}$, and the second on the left by $\mathbf{a}^{\mathrm{T}}$.

$$\mathbf{b}^{\mathrm{T}}\mathbf{K}\mathbf{a} = \lambda_1\mathbf{b}^{\mathrm{T}}\mathbf{M}\mathbf{a}, \qquad \mathbf{a}^{\mathrm{T}}\mathbf{K}\mathbf{b} = \lambda_2\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{b}. \tag{23}$$

Take the transpose of the entire first of Eqs. (23):

$$\left(\mathbf{b}^{\mathrm{T}}\mathbf{K}\mathbf{a}\right)^{\mathrm{T}} = \lambda_1\left(\mathbf{b}^{\mathrm{T}}\mathbf{M}\mathbf{a}\right)^{\mathrm{T}}. \tag{24}$$

The transpose of a product of matrices equals the product of the transposes in the reverse order; also, $(\mathbf{b}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{b}$, so Eq. (24) can be written

$$\mathbf{a}^{\mathrm{T}}\mathbf{K}^{\mathrm{T}}\mathbf{b} = \lambda_1\mathbf{a}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{b}. \tag{25}$$

If $\mathbf{K}$ and $\mathbf{M}$ are symmetric, as in our example, then $\mathbf{K}^{\mathrm{T}} = \mathbf{K}$ and $\mathbf{M}^{\mathrm{T}} = \mathbf{M}.$ (We'll discuss this property later.) Then Eq. (25) can be written.

$$\mathbf{a}^{\mathrm{T}}\mathbf{K}\mathbf{b} = \lambda_1\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{b}. \tag{26}$$

Finally, we subtract Eq. (26) from the second of Eqs. (23):

$$(\lambda_2 - \lambda_1)\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{b} = 0. \tag{27}$$

Thus if $\lambda_2 \neq \lambda_1$, then $\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{b} = 0$, and the eigenvectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal in the generalized sense we have defined. Similarly, any pair of eigenvectors with unequal eigenvalues are orthogonal in this sense.

Initial Conditions

We can now use the orthogonality properties of the normalized eigenvectors to work out relations to find the constants $C_1,\ C_2,\ C_3,\ \cdots$ and $\varphi_1,\ \varphi_2,\ \varphi_3,\ \cdots$ in Eq. (12) if we are given the initial values of the coordinates and velocities, which we denote by $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$, respectively. At time $t = 0$, Eq. (12) becomes

$$\mathbf{x}(0) = C_1\mathbf{a}\cos\varphi_1 + C_2\mathbf{b}\cos\varphi_2 + C_3\mathbf{c}\cos\varphi_3 + \cdots. \tag{28}$$

Similarly, taking the time derivative of Eq. (12) and setting $t = 0$, we find

$$\dot{\mathbf{x}}(0) = -C_1\omega_1\mathbf{a}\sin\varphi_1 - C_2\omega_2\mathbf{b}\sin\varphi_2 - C_3\omega_3\mathbf{c}\sin\varphi_3 - \cdots. \tag{29}$$

Now see what happens when we multiply Eq. (28) on the left by $\mathbf{a}^{\mathrm{T}}\mathbf{M}.$ We get

$$\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{x}(0) = C_1\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{a}\cos\varphi_1 + C_2\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{b}\cos\varphi_2 + C_3\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{c}\cos\varphi_3 + \cdots. \tag{30}$$

The matrix product in the first term on the right is unity, because of the normalization of the eigenvectors, and all the other matrix products on the right are zero because of orthogonality . So Eq. (30) becomes simply

$$\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{x}(0) = C_1\cos\varphi_1. \tag{31}$$

The same thing happens when we multiply Eq. (29) on the left by $\mathbf{a}^{\mathrm{T}}\mathbf{M}$

$$\mathbf{a}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{x}}(0) = -C_1\omega_1\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{a}\sin\varphi_1 - C_2\omega_2\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{b}\sin\varphi_2 - C_3\omega_3\mathbf{a}^{\mathrm{T}}\mathbf{M}\mathbf{c}\sin\varphi_3 + \cdots. \tag{32}$$

Using the normalization and orthogonality conditions, we get

$$\mathbf{a}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{x}}(0) = -C_1\omega_1\sin\varphi_1. \tag{33}$$

Now it is straightforward to solve Eqs. (31) and (33) simultaneously to obtain C_1 and φ_1. And all the other C's and φ's can be obtained the same way by multiplying Eqs. (28) and (29) by **b, c,** $\cdots$.

Diagonalization of K and M Matrices

The eigenvectors and eigenvalues can be used to construct the normal-coordinate transformation; this transformation converts both **K** and **M** into *diagonal* matrices. From Eqs. (11),

$$\mathbf{Ka} = \lambda_1\mathbf{Ma}, \qquad \mathbf{Kb} = \lambda_2\mathbf{Mb}, \qquad \mathbf{Kc} = \lambda_3\mathbf{Mc}, \quad \cdots \quad . \tag{34}$$

We can combine these equations into a single matrix equation. Let $a_1, a_2, a_3, \ldots$ be the elements of the normalized eigenvector **a**, and so on; then define a matrix **A** consisting of the normalized eigenvectors as columns; that is.

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \qquad \text{(for two degrees of freedom)}, \tag{35}$$

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \qquad \text{(for three degrees of freedom)}, \tag{36}$$

and so on. (From here on we assume two degrees of freedom; the generalization to three or more degrees of freedom will be clear.)

We can combine Eqs. (34) as

$$\mathbf{K}\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \mathbf{M}\begin{pmatrix} \lambda_1 a_1 & \lambda_2 b_1 \\ \lambda_1 a_2 & \lambda_2 b_2 \end{pmatrix}. \tag{37}$$

Also note that

$$\begin{pmatrix} \lambda_1 a_1 & \lambda_2 b_1 \\ \lambda_1 a_2 & \lambda_2 b_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{38}$$

Now we define a diagonal matrix **L** as

$$\mathbf{L} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{39}$$

Then

$$\begin{pmatrix} \lambda_1 a_1 & \lambda_2 b_1 \\ \lambda_1 a_2 & \lambda_2 b_2 \end{pmatrix} = \mathbf{AL},$$

and Eq. (37) can be written as

$$\mathbf{KA} = \mathbf{MAL}. \tag{40}$$

Now we multiply this equation on the left by $\mathbf{A}^{\mathrm{T}}$:

$$\mathbf{A}^{\mathrm{T}}\mathbf{KA} = \mathbf{A}^{\mathrm{T}}\mathbf{MAL}. \tag{41}$$

Consider the product $\mathbf{A}^{\mathrm{T}}\mathbf{MA}$. In general, this is equal to

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.$$

We see that the first row of $\mathbf{A}^{\mathrm{T}}$ and the first column of $\mathbf{A}$ give the normalization condition for the eigenvector **a.** The first row of $\mathbf{A}^{\mathrm{T}}$ and the second column of $\mathbf{A}$ give the orthogonality relation for **a** and **b**, and so on. Thus the diagonal elements of the product $\mathbf{A}^{\mathrm{T}}\mathbf{MA}$ are all 1, and the off-diagonal elements are all zero. That is, $\mathbf{A}^{\mathrm{T}}\mathbf{MA}$ is equal to the identity (unit) matrix, denoted by **I**.

It follows from this result that

$$\left(\mathbf{A}^{\mathrm{T}}\mathbf{MA}\right)\Lambda = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{42}$$

and, from Eq. (41),

$$\mathbf{A}^{\mathrm{T}}\mathbf{KA} = \Lambda. \tag{43}$$

(Note that Λ is always diagonal, and that the diagonal elements are the eigenvalues.)

Thus we have the fairly astonishing result that the transformation $\mathbf{A}^{\mathrm{T}}\mathbf{MA}$ turns **M** into the identity matrix **I**, and it turns **K** into a matrix $\mathbf{A}^{\mathrm{T}}\mathbf{KA} = \mathbf{L}$ that is always diagonal, with diagonal elements equal to the eigenvalues.

Example

For the same example we've been discussing all along,

$$\mathbf{a} = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{A} = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{44}$$

$$\mathbf{A}^{\mathrm{T}}\mathbf{MA} = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\frac{1}{\sqrt{2m}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{45}$$

$$\mathbf{A}^{\mathrm{T}}\mathbf{KA} = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k + k' & -k' \\ -k' & k + k' \end{pmatrix}\frac{1}{\sqrt{2m}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \dfrac{k}{m} & 0 \\ 0 & \dfrac{k + 2k'}{m} \end{pmatrix}. \tag{46}$$

You should verify Eqs. (45) and (46) by carrying out the matrix multiplication, both by hand and using Maple.

Normal Coordinate Transformation

The normal coordinate transformation is now simply $\mathbf{x} = \mathbf{A}\mathbf{q}$. We substitute this into the equations of motion, Eq. (4):

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}, \qquad \mathbf{M}\mathbf{A}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{A}\mathbf{q}. \tag{47}$$

Multiply on the left by $\mathbf{A}^{\mathrm{T}}$:

$$\mathbf{A}^{\mathrm{T}}\mathbf{M}\mathbf{A}\ddot{\mathbf{q}} = -\mathbf{A}^{\mathrm{T}}\mathbf{K}\mathbf{A}\mathbf{q}. \tag{48}$$

But, as we showed above,

$$\mathbf{A}^{\mathrm{T}}\mathbf{M}\mathbf{A} = \mathbf{I} \qquad \text{and} \qquad \mathbf{A}^{\mathrm{T}}\mathbf{K}\mathbf{A} = \mathbf{L}. \tag{49}$$

So Eq. (48) becomes

$$\ddot{\mathbf{q}} = -\Lambda\mathbf{q}, \qquad \text{or} \qquad \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = -\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = -\begin{pmatrix} \lambda_1 q_1 \\ \lambda_1 q_2 \end{pmatrix}. \tag{50}$$

We see that the equations for the q's are *decoupled*; each equation contains only one q:

$$\ddot{q}_1 = -\lambda_1 q_1 = -\omega_1^2 q_1, \qquad \ddot{q}_2 = -\lambda_2 q_2 = -\omega_2^2 q_2, \tag{51}$$

and so forth. Each q corresponds to a single normal mode.

Symmetry of M and K Matrices

In our example problem, both **M** and **K** happen to be *symmetric*. When they are not, our orthogonality proof is not valid, and Eqs. (26) and (27) aren't correct. When the equations of motion are obtained from Lagrange's equations, it can be shown that **M** and **K** are *always* symmetric. But when we start with $\Sigma F = ma$, the resulting matrices may or may not be symmetric, depending on our choice of coordinates.

Suppose, for instance, that for our example system we let x_1 be the displacement of the left mass from equilibrium and let x_2 be the elongation of the spring k'. Then the displacement of the right mass from equilibrium is $x_1 + x_2$ and its acceleration is $\ddot{x}_1 + \ddot{x}_2$ You can verify that the equations of motion, in terms of these coordinates, are

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + k'x_2, \\ m(\ddot{x}_1 + \ddot{x}_2) &= -kx_1 - (k + k')x_2. \end{aligned} \tag{52}$$

In matrix form,

$$\begin{pmatrix} m & 0 \\ m & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k & -k' \\ k & k + k' \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{53}$$

The **M** and **K** matrices are

$$\mathbf{M} = \begin{pmatrix} m & 0 \\ m & m \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} k & -k' \\ k & k + k' \end{pmatrix}. \tag{54}$$

Note that neither of these is symmetric. The secular equation is

$$|\mathbf{K} - \lambda\mathbf{M}| = \begin{vmatrix} k - \lambda m & -k' \\ k - \lambda m & k + k' - \lambda m \end{vmatrix} = 0. \tag{55}$$

We leave it as an exercise for the reader to show that the eigenvalues are again

$$\lambda_1 = \frac{k}{m}, \qquad \lambda_2 = \frac{k + 2k'}{m}, \tag{56}$$

and the normalized eigenvectors are

$$\mathbf{a} = \frac{1}{\sqrt{m}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{b} = \frac{1}{\sqrt{3m}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \mathbf{A} = \frac{1}{\sqrt{m}}\begin{pmatrix} 1 & 1/\sqrt{3} \\ 0 & -2/\sqrt{3} \end{pmatrix}. \tag{57}$$

You should check that again the operation $\mathbf{A}^{\mathrm{T}}\mathbf{M}\mathbf{A}$ reduces $\mathbf{M}$ to the identity matrix $\mathbf{I}$, and that

$$\mathbf{A}^{\mathrm{T}}\mathbf{K}\mathbf{A} = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{58}$$

You can also check the orthogonality of the eigenvectors; in this case they are *not* orthogonal because **M** and **K** aren't symmetric. However, we can reformulate the problem by adding and subtracting the equations of motion, Eqs. (52), to obtain **M** and **K** matrices that *are* symmetric. Here's how we do it. We can add the two original equations of motion, Eqs. (52), to obtain

$$m(2\ddot{x}_1 + \ddot{x}_2) = -2kx_1 - kx_2. \tag{59}$$

Now we take this equation and the second of Eqs. (52) to be the be the basic equations of motion:

$$\begin{aligned} m(2\ddot{x}_1 + \ddot{x}_2) &= -2kx_1 - kx_2, \\ m(\ddot{x}_1 + \ddot{x}_2) &= -kx_1 - (k + k')x_2. \end{aligned} \tag{60}$$

For these equations, the **M** and **K** matrices are

$$\mathbf{M} = \begin{pmatrix} 2m & m \\ m & m \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} 2k & k \\ k & k + k' \end{pmatrix}. \tag{61}$$

Now both **M** and **K** are symmetric. The secular determinant is

$$\begin{vmatrix} 2k - 2\lambda m & k - \lambda m \\ k - \lambda m & k + k' - \lambda m \end{vmatrix} = 0. \tag{62}$$

You are invited to complete this analysis by (1) showing that the roots of the secular equation are the same as before, (2) obtaining the normalized eigenvector for each eigenvalue, and (3) showing that the eigenvectors are orthogonal. Have fun!