7 Mapping and Chaos Fall 2003

When a periodically varying driving force is applied to a damped oscillating system, is the resulting forced oscillation always periodic, with the same period? In the case of the damped harmonic oscillator (with restoring force $-kx$ and damping force $-bv$), the answer is *yes*. When a sinusoidally varying driving force is applied, the resulting forced oscillation is also sinusoidal, with the same frequency as the driving force. There may also be some transient motion, depending on initial conditions, but this dies out with time, leaving a forced-oscillation motion whose frequency, amplitude, and phase are completely predictable. The motion can be predicted as far in the future as we wish.

As we have already seen, the harmonic oscillator is a very special kind of oscillating system. When the restoring and damping forces are *nonlinear* functions of x and v, the period of *free* oscillation depends on amplitude, and the *forced* oscillations produced by a periodic driving force are *not* necessarily periodic. In some cases the motion may approach a limit cycle, but in others it may be non-periodic and *chaotic*. In this case it is impossible to predict the long-term behavior, not only because the equations of motion have to be solved numerically, but also (and more importantly) because the motion is extremely sensitive to initial conditions. No matter how great the precision of the initial conditions, this precision is lost in a relatively short time, and the motion becomes unpredictable.

This is a little startling. We're accustomed to thinking of Newtonian mechanics as a *deterministic* theory. Given initial conditions and a description of the forces, we expect to be able to predict the resulting motion into the indefinite future, with as great precision as we want. How is it possible that chaotic, unpredictable motion can occur in a deterministic system? This and similar questions have been studied intensively only in the past 30 years or so; following is a brief sketch of some of the analytical methods that have been used.

Periodic Forcing Function

Suppose we have a damped oscillating system with one degree of freedom (such as a point mass moving along a straight line). A periodic driving force with period T is applied. The force need not be sinusoidal, but it must be exactly periodic. The vector field plot for the system changes with time because of the time-dependent force. But because the force is periodic, the field plot at some initial time t_1 looks exactly like the field plot one period later, at time $t_1 + T$.

In general, if at time t_1 the phase point is at (x_1, v_1), then one period later, at time $t_1 + T$ it will be located at a different point (x_2, v_2). But if the phase point is at (x_1, v_1) at any time that is an integer number n of periods after t_1, that is, at any time with the form $t_1 + nT$, then it must be at point (x_2, v_2) one period later, at time $t_1 + (n+1)T$. The reason this must be true is that the vector field pattern is the same at times t, $t + T$,

$t + 2T$, and so on. The vector field changes in a periodic way, and it goes through the same sequence of patterns during the intervals between times t and $t + T$, between $t + T$ and $t + 2T$, and so on.

A simple way to think about such motions is to focus not on the entire phase trajectory but on the location of the phase point at a series of particular times separated by one period. If the phase point is at (x_1, v_1) at some time t_1, it will be at (x_2, v_2) one period later, at time $t_1 + T$. For any given value of t_1, (x_2, v_2) is a function of (x_1, v_1) often called a *mapping* of (x_1, v_1) to (x_2, v_2). Symbolically, $(x_2, v_2) = f(x_1, v_1)$. If we knew the details of this mapping, we could predict the motion for long times by iterating this mapping for as many periods as we want: $(x_3, v_3) = f(x_2, v_2)$, and so on.

Pursuing this idea, suppose we start at phase point (x_1, v_1) at time t_1, and then note the location of the phase point at the end of each period following t_1. That is, instead of plotting a continuous curve in the phase plane, we plot a series of points, representing the phase point at successive times t_1, $t_1 + T$, $t_1 + 2T$, $t_1 + 3T$, and in general $t_1 + nT$, where n is a positive integer. From this pattern we can see the general nature of the motion. If successive points converge toward one location, we have a *limit cycle*. In that case, (x_{n+1}, v_{n+1}) approaches (or converges toward) (x_n, v_n) after a long time (i.e., large n). The limiting motion is then a repeating cycle with the same period T as the driving force, and the point of convergence is an *attractor*.

It may also happen that after a long time the phase point settles down to a pattern of *alternation* between two locations,. This is again a limit cycle, but with period $2T$ and *two* attractors. This phenomenon is called *period doubling*. There may also be cycles with longer periods that are larger integer multiples of T. Such a motion would show up as several points in the phase plane, with a definite pattern of repetition.

Finally, if the motion is *chaotic*, the pattern of points never repeats, corresponding to a phase trajectory that goes on endlessly, never repeating itself. In this case, no point ever falls exactly on the location of a previous point; if it did, the motion would repeat exactly, over and over again, the pattern seen between the two identical points.

The Logistic Map

In the preceding discussion, it is usually impossible to represent the mapping $(x_2, v_2) = f(x_1, v_1)$ as an actual function, but the discussion may help to show why the study of mappings is crucial to the understanding of the response of a nonlinear system to periodic driving forces. To understand further how mappings can be studied, we focus on a simpler problem, the behavior of a *one-dimensional* mapping that maps a point x_1 on a line to another point x_2 on the line, that is, $x_2 = f(x_1)$.

A mapping that has been extensively studied is the *logistic map*. It was originally introduced as a model of population growth or decay for a particular species of animal. Suppose P_n is the population in year n. Assume that the population P_{n+1} (in the year

$(n + 1)$ differs from P_n because of a number of new births, which we assume to be proportional to P_n, and because of deaths due to limited food supply, assumed proportional to P_n^2. The appropriate equation is then

$$P_{n+1} = a\,P_n - b\,P_n^2 \tag{1}$$

where a and b are positive constants and n is a non-negative integer..

It is convenient to change the variable from P_n to a normalized variable x_n, limited to the range $0 \le x_n \le 1$. The usual form of the *logistic map* or *logistic equation* is

$$x_{n+1} = a\,x_n(1 - x_n)\,, \tag{2}$$

where a is a positive constant. Considering the function $a\,x\,(1 - x)$, we note that its maximum value in the interval $0 \le x \le 1$ is $a/4$. Thus to insure that $0 \le x_{n+1} \le 1$ whenever $0 \le x_n \le 1$, for all n, we restrict a to the range $0 \le a \le 4$.

The simple and innocent-appearing mapping represented by Eq. (2) turns out to have an amazingly rich variety of properties, some of them quite unexpected. Here are some simple questions we can ask. Can the population ever be *constant* from one year to the next, so that $x_{n+1} = x_n$? If it is *not* constant, can it approach a limit after many years, so that x_n approaches a *limit* at large n? If so, does the limit depend on the initial value x_o? Can the variation of x_n be random, with *no* repeating pattern?

As a preliminary exploration, let's start with a few numerical experiments. We'll choose a value of the constant a and an initial x, which we'll call x_o, and compute a sequence of values of x_n for increasing integer values of n. This is easy to do using Maple. Suppose we choose the values $a = 0.5$ and $x_o = 0.5$; we can use the Maple code

```
restart;
Digits := 5;
a := 0.5;   x[0] := 0.5;   N := 10;
for n from 0 to N do
x[n + 1] := evalf(a*x[n]*(1 – x[n]));
end do;
```

Maple gives us this sequence of values of x:

$x_1 = 0.12500$	$x_6 = 0.0030885$
$x_2 = 0.54688$	$x_7 = 0.0015394$
$x_3 = 0.025849$	$x_8 = 0.00076851$
$x_4 = 0.012590$	$x_9 = 0.00038396$
$x_5 = 0.0062157$	$x_{10} = 0.000019191$

Clearly, this species is headed for extinction; at large n, x_n approaches zero. In fact, we note that if we had taken $x_o = 0$, *all* the subsequent x's would have been zero.

With $a = 2.5$ and $x_o = 0.5$, we get this sequence:

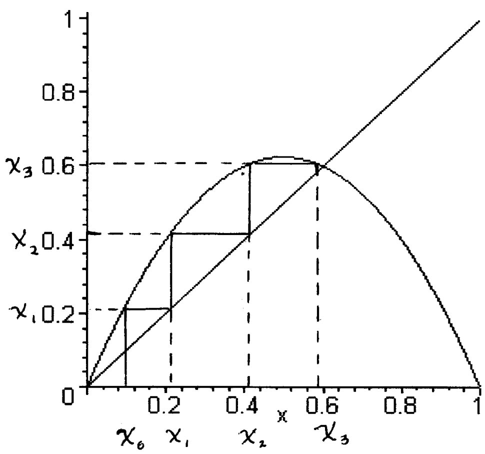
$x_1 = 0.62500$	$x_6 = 0.59917$
$x_2 = 0.58594$	$x_7 = 0.60040$
$x_3 = 0.60652$	$x_8 = 0.59980$
$x_4 = 0.59663$	$x_9 = 0.60010$
$x_5 = 0.60167$	$x_{10} = 0.59993$

The final limiting value is clearly 0.60000, and the values of x_n alternate on both sides of the final value as the limit is approached.

We could have predicted the existence of this limit without all the arithmetic. When the limit is reached, we must have $x_{n+1} = x_n$ for all n. When $a = 2.5$, the limiting value of x_n must satisfy the equation

$$x = ax(1-x)\,. \tag{3}$$

When $a = 2.5$, this becomes $x = 2.5x(1-x)$. We invite you to verify that the roots of this quadratic equation are $x = 0$ and $x = 0.6$. So we can see why $x = 0.6$ is a limit point, and we can call it an *attractor*. This value is independent of the value of x_o We'll return later to the significance of the other root, $x = 0$.



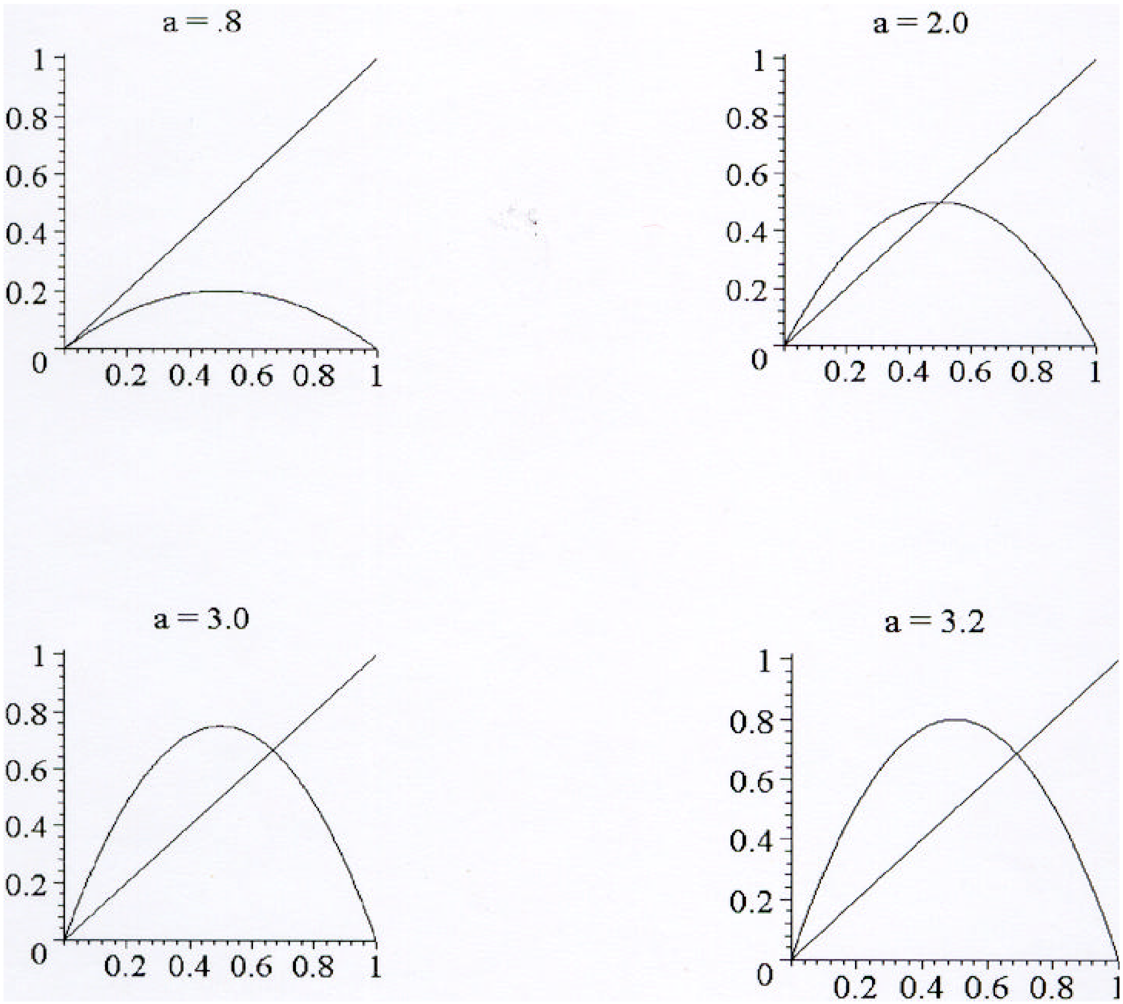
A graphical representation of the logistic map offers additional general insight. The function $y = ax(1-x)$ is an inverted parabola with x-intercepts at $x = 0$ and $x = 1$. Its maximum value, occuring at $x = 1/2$, is $a/4$, and its slope at $x = 0$ is a. (We invite you to verify these statements.) The function $y = x$ is a straight line at 45^o to the $+x$ axis. Any possible attractors must occur at points where $x = ax(1-x)$, that is, where these two functions are equal, and the parabola and the straight line intersect.

The point $x = 0$ is *always* an intersection point. When $0 \le a \le 1$, the slope of the parabola at $x = 0$ is *less than* that of the line, so there can be no other intersection point in the relevant interval $0 \le x \le 1$. When $1 < a \le 4$, there is a second intersection point at $x = 1 - 1/a$. In the second example above (shown in the figure), $a = 2.5$ and $x = 1 - 1/2.5 = 0.6$, as the numerical calculations also show.

To construct a sequence of x's, choose an initial value x_o; in the figure, we have taken $x_o = 0.1$. Draw a vertical line up to the parabola; its height is x_1. From this point, draw a horizontal line over to the 45^o line. The length of this line (from the vertical axis) is also x_1. So a vertical line through this point, from the horizontal axis up to the parabola has length x_2. Draw a horizontal line from the parabola to the 45^o line, then a vertical line (representing x_3) from the horizontal axis to the parabola, and so on. We can see that the sequence of x's approaches the attractor, $x = 0.6$. We can also see that the point

$x = 0$ is *not* an attractor; successive x's move farther and farther from $x = 0$, and it is a *repeller*, not an attractor. (If the above description of this construction isn't clear, keep going over it until you have an "aha!" moment.)

Below are graphs for several values of a. We invite you to practice this construction on the graphs.



Now it's time for another numerical experiment. We take $a = 3.1$ and $x_o = 0.5$; the Maple calculation gives us the sequence

$x_1 = 0.77500$	$x_6 = 0.55319$
$x_2 = 0.54056$	$x_7 = 0.76623$
$x_3 = 0.76988$	$x_8 = 0.55527$
$x_4 = 0.54920$	$x_9 = 0.76551$
$x_5 = 0.76749$	$x_{10} = 0.55647$

What's happening? The sequence is clearly not converging to a single value, but instead it is converging toward an alternation between two values, at around 0.557 and 0.765.

Unlike the preceding examples, with a single attractor such that in the limit of large n we had $x_{n+1} = x_n$, we have two attractors, and two different sequences, such that in the limit of large n we have $x_{n+2} = x_n$ for each n. The conditions for this to happen can be stated as:

$$
\begin{aligned}
x_{n+1} &= ax_n(1 - x_n), \\
x_{n+2} &= ax_{n+1}(1 - x_{n+1}) \\
&= a\big[ax_n(1 - x_n)\big]\big\{1 - \big[ax_n(1 - x_n)\big]\big\}
\end{aligned}
$$

When we set $x_{n+2} = x_n$, the result is a fourth-degree equation for x. One form is

$$x = -a^2 x(x - 1)(ax^2 - ax + 1)$$

Substituting the value $a = 3.1$ and solving the resulting equation numerically using Maple, we obtain the following roots (to five significant figures):

0, 0.55801, 0.67742, 0.76457.

We see that the second and fourth roots are the attractors we found previously, while the first and third are repellers. Note that *all* these numerical values are independent of x_0.

This splitting of one attractor into two is called a *bifurcation*. Further numerical experiments show that it occurs at exactly $a = 3$. When $1 < a < 3$, there is a single attractor, but when a is slightly greater than 3, there are two.

Continuing our numerical experiments, we increase a further and observe the results. It's easier to see what's happening if we represent the sequence of x_n's as a graph, using the following Maple code:

```
restart;
a ;= 3.2;    x[0] := 0.5;    N := 20;
for n from 0 to N do
x[n + 1] := a*x[n]*(1 – x[n]);
end do;
pointlist := seq([n, x[n]], n = 0..N):
plot([pointlist], style = point, symbol = cross, color = black);
```

This code plots a sequence of points (n, x_n), each represented by a small cross, for any chosen values of x_0 and a. We can change the total number N of points to show more clearly how quickly or slowly the sequence converges to the attractors.

At approximately $a = 3.449490$ *another* bifurcation occurs; for values of a slightly larger than this, there are *four* attractors. At $a = 3.35444090$ we find still another bifurcation, beyond which there are eight attractors. Further increases in a reveal an infinite series of bifurcations. We can show in tabular form where the bifurcations occur. We number the bifurcations with an index k. After the first bifurcation $(k = 1)$ there are 2 $(= 2^1)$ attractors; after the second $(k = 2)$ we find four $(= 2^2)$, after the third $(k = 3)$, eight $(= 2^3)$, and so on. We denote the corresponding values of a by a_k.

k	a_k	2^k
1	3.000000	2
2	3.449490	4
3	3.544090	8
4	3.564407	16
5	3.568759	32

This is amazing enough, but it gets even better. We note that with increasing k, the bifurcation points become closer together. Indeed, they approach a limit, which we may call a_∞, at large k. It turns out that $a_\infty = 3.569946$.

When $a > a_\infty$, in general there are *no* attractors; instead, the sequence of x_n*'s* fills an entire range of values of x in a random and chaotic manner. There are, however, some small "islands" (ranges of values of a) where there are period-three attractors. These are called strange attractors, referring to the fact that they are not well understood.

The decreasing intervals between successive a_k's can be described by the ratio

$$\frac{a_k - a_{k-1}}{a_{k+1} - a_k}. \tag{4}$$

It turns out that this ratio approaches a limit as $k \to \infty$; we denote this limit by δ:

$$d = \lim_{k\to\infty} \frac{a_k - a_{k-1}}{a_{k+1} - a_k} = 4.669201609\,. \tag{5}$$

This number, called the *Feigenbaum constant* after its discoverer, is obtained by numerical calculations of the same sort we have been describing. It can't be computed from any known formula.

Now, finally, here's something that's *really* remarkable. We've discussed the Feigenbaum constant in the specific context of the logistic map $x_{n+1} = a\,x_n(1 - x_n)$. But it turns out that for *any* mapping function $x_{n+1} = f(x_n)$ that is continuous and concave downward, with only one maximum in the interval $0 \le x \le 1$, the limit defined in Eq. (5) has the same numerical value!! This is a truly amazing result that is still not well understood.

This discussion has shown that even a very simple-looking mapping such as Eq. (2) can exhibit a remarkable richness of behavior. Finally, we return to the two-dimensional phase space with which this whole discussion began. If chaos can occur with the simple one-dimensional map of Eq. (2), it shouldn't be surprising that it can occur also with the much more complex two dimensional maps $(x_2, v_2) = f(x_1, v_1)$ corrsponding to forced oscillations with periodic driving forces.

For more complex systems, with n coordinates (i.e., n degrees of freedom), the phase space has $2n$ dimensions. The coordinates of the phase space are often taken to be the generalized coordinates and momenta that play a central role in the Lagrangian and Hamiltonian formulations of classical mechanics. If the coordinates are Cartesian coordinates of N particles, the coordinates of the phase space are usually taken to be the coordinates x_n and the momentum components $p_n = m_n v_n$, where n ranges from zero to $3N$.