33-232 **Final Exam** **December 14, 2000**

1. (40 points = 8 + 8 + 8 + 8 + 8) Name Young

1	
2	
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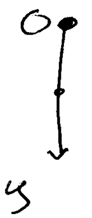
6	
total	

A ball with mass m is dropped vertically downward from the top of a tall building, in a uniform gravitational field g; with zero initial speed.

a) If air resistance can be neglected, derive expressions for the velocity and position as functions of time. Be sure to define your coordinate system carefully, and show the steps in your derivation, starting with basic physical laws.



$$F_y = mg = m\frac{d^2y}{dt^2} \leftarrow \text{integrate}$$

$$\frac{dy}{dt} = v = v_0 + gt \leftarrow$$

$$y = y_0 + v_0 t + \frac{1}{2}gt^2$$

$$y_0 = 0, \quad v_0 = 0$$

$$\boxed{v = gt \qquad y = \frac{1}{2}gt^2}$$

b) Now suppose there is an air-resistance force F with magnitude proportional to the mass m and to the speed $|v|$ of the ball, i.e., $|F| = bm|v|$, where b is a constant. Derive an expression for the velocity v of the ball as a function of time.

$$\cancel{m}\frac{dv}{dt} = \cancel{m}g - b\cancel{m}v$$

$$\frac{dv}{dt} + bv = g \qquad \text{particular sol.: } v = \frac{g}{b}$$

$$\text{complem. sol: } \quad v = \text{const.}\, e^{-bt}$$

$$v = \text{const}\, e^{-bt} + \frac{g}{b} \qquad v(0) = 0 \Rightarrow 0 = \text{const} + \frac{g}{b}$$

$$v = \frac{g}{b}\left(1 - e^{-bt}\right)$$

(continued)

1. (continued)

c) Show that in the limit as $b \to 0$, your result for part (b) becomes the result for v in part (a). (<u>Hint</u>: You can't just set $b = 0$ because the result is indeterminate. Instead, expand your result for part (b) in a power series.)

$$e^{-bt} = 1 - bt + \frac{1}{2}b^2t^2 + \cdots$$

$$v = \frac{g}{b}\left[\cancel{1} - (\cancel{1} - bt + \tfrac{1}{2}b^2t^2 + \cdots\right]$$

$$= \frac{g}{b}\left[bt - \tfrac{1}{2}b^2t^2 + \cdots\right]$$

$$= gt - \tfrac{1}{2}bgt^2 + \cdots$$

as $b \to 0$, all terms except first $\to 0$

and $v \to gt$

d) Derive an expression for the *terminal velocity* of the ball in two ways: First, without reference to the result of part (b), and second by using the result of part (b).

$m\frac{dv}{dt} = mg - bmv$, as terminal velocity is reached, $\frac{dv}{dt} = 0 \Rightarrow mg - bmv = 0$,

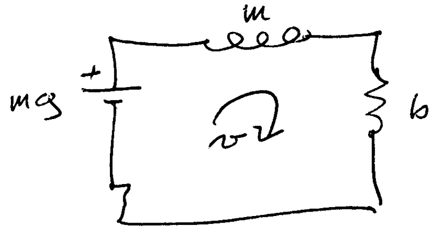
(1) $v = \frac{mg}{mb} = \frac{g}{b}$

(2) as $t \to \infty$, $e^{-bt} \to 0$, $v \to \frac{g}{b}$

(continued)

1. (continued)

e) Devise an electric circuit that is the analog of this mechanical system. I.e., find a circuit for which the differential equation has the same form as for the mechanical system. State clearly the electrical quantity that corresponds to each mechanical quantity in the original problem.



$$m\frac{dv}{dt} \cancel{+} mg + bv = 0$$

$$mg - m\frac{dv}{dt} - bv = 0$$

mg corresponds to battery voltage $\mathcal{E}$

m corresponds to inductance L

b " " resistance R

v " " current i

2. (35 points = 5 + 10 + 10 + 10)

A damped harmonic oscillator is constructed using a spring with force constant 1200 N/m, an object with mass 3.00 kg, and a shock absorber that applies a damping force with magnitude bv, where b is a constant whose value is not given. A sinusoidal driving force $F = F_0 \cos \omega t$ is applied. As ω is varied (with F_0 kept constant), the amplitude of the resulting forced oscillation also varies. Its greatest value is 0.0200 m; this value occurs when $\omega = 20.0\ s^{-1}$. When $\omega = 22.0\ s^{-1}$, the amplitude is 0.0141 m.

a) Discuss briefly why the "small-damping" approximation is appropriate.

Resonance curve is sharply peaked. When ω differs from peak value by 10%, amplitude of forced oscillation drops to $\frac{1}{\sqrt{2}}$ of max. value

b) Determine the numerical value of the constant b, with correct units.

$$A' = \frac{F_0/2m\omega_0}{\sqrt{(\omega-\omega_0)^2+\gamma^2}}$$

$$\frac{A'(\omega_0)}{A'(\omega)} = \sqrt{2} = \frac{\sqrt{(20s^{-1}-22s^{-1})^2+\gamma^2}}{\sqrt{0+\gamma^2}}$$

$$2 = \frac{4s^{-2}+\gamma^2}{\gamma^2}$$

$$\gamma = 2s^{-1} = \frac{b}{2m}$$

$$b = 2m\gamma = 2(3.0\ kg)(2s^{-1}) = \boxed{12\ kg/s}$$

(continued)

2. (continued)

c) Determine the numerical value of the constant F_o, with correct units.

$$A'(\omega_0) = \frac{F_0/(2m\omega_0)}{\sqrt{0+\gamma^2}} = \frac{F_0}{2m\gamma\omega_0}$$

$$\begin{aligned} F_0 &= A'(\omega_0)\, 2m\gamma\omega_0 \\ &= (0.02\,\text{m})(\cancel{20\,\text{s}^{-1}})(2)(3.0\,\text{kg})(2\,\text{s}^{-1})(20\,\text{s}^{-1}) \\ &= 4.80\ \text{kg m/s}^2 = \boxed{4.90\,\text{N}} \end{aligned}$$

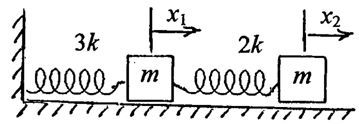
d) Suppose we now remove the driving force, give the mass some displacement x_o, and release it. Determine the time required for the amplitude of the resulting oscillations to decrease to $1/e$ of its initial value.

$$A = A_0 e^{-\gamma t}$$

$$\gamma t = 1 \Rightarrow t = \frac{1}{\gamma} = \frac{1}{2\,\text{s}^{-1}} = \boxed{\tfrac{1}{2}\,\text{s}}$$

3. (40 points = 10 + 10 + 10 + 10)

a) For the system shown, obtain the $\Sigma F = ma$ equations.



$$m\ddot{x}_1 = -3kx_1 + 2k(x_2 - x_1)$$

$$= -5kx_1 + 2kx_2$$

$$m\ddot{x}_2 = -2k(x_2 - x_1) = 2kx_1 - 2kx_2$$

b) Express these equations as a matrix equation, and obtain the **M** and **K** matrices.

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} 5k & -2k \\ -2k & 2k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad K = \begin{pmatrix} 5k & -2k \\ -2k & 2k \end{pmatrix}$$

(continued)

3. (continued)

c) Use the $\mathbf{M}$ and $\mathbf{K}$ matrices to derive expressions for the angular frequencies of all the normal modes. <u>Hint</u>: The frequencies are all integer multiples of $\sqrt{k/m}$.

$$|K - \lambda M| = 0 \qquad (\lambda = \omega^2$$

$$\begin{vmatrix} 5k - \lambda m & -2k \\ -2k & 2k - \lambda m \end{vmatrix} = 0$$

$$(5k - \lambda m)(2k - \lambda m) - 4k^2 = 0$$

$$10k^2 - 7k\lambda m + \lambda^2 m^2 - 4k^2 = 0$$

$$\lambda^2 m^2 - 7k\lambda m + 6k^2 = 0 = (\lambda m - k)(\lambda m - 6k)$$

$$\lambda = \frac{k}{m}, \ \frac{6k}{m} \qquad \boxed{\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{6k}{m}}}$$

d) If the left-hand mass is given an initial displacement of 0.050 m, what initial displacement should the right-hand mass be given if the resulting motion includes only the normal mode with the smaller frequency? (Assume both masses are given zero initial velocity.)

$$\begin{pmatrix} 5k - \lambda m & -2k \\ -2k & 2k - \lambda m \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \qquad \text{for } \lambda = \frac{k}{m},$$

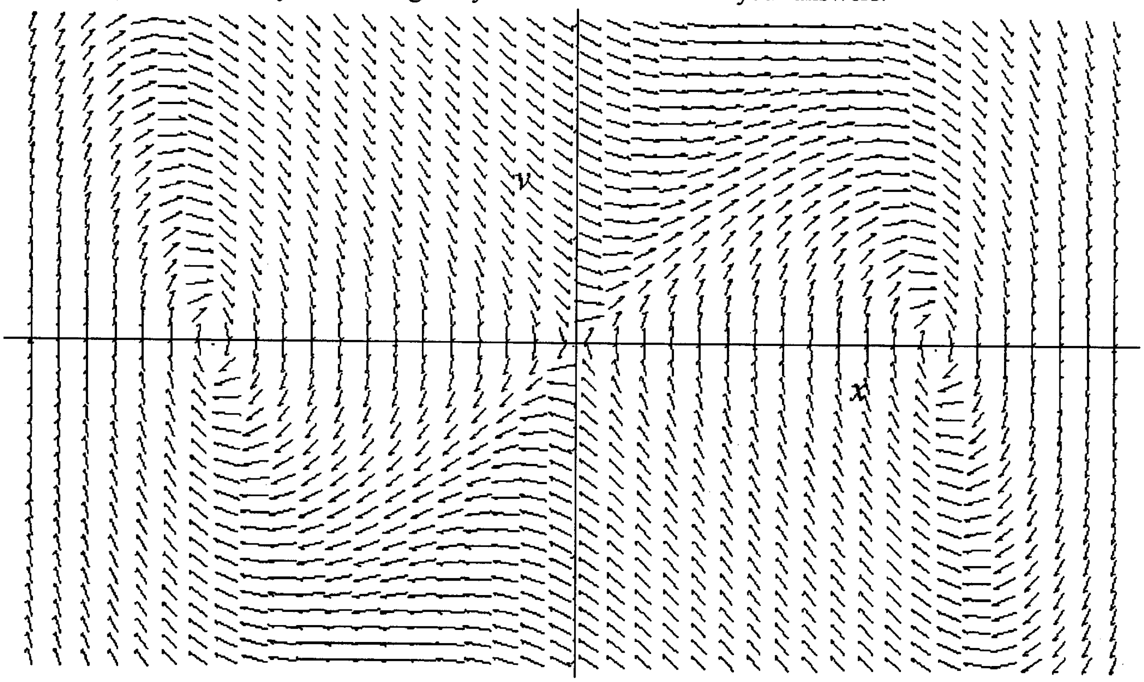
$$\begin{pmatrix} 5k - k & -2k \\ -2k & 2k - k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \qquad \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$4A_1 - 2A_2 = 0 \qquad A_2 = 2A_1$$

$$\text{for } A_1 = 0.050 \text{ m}, \quad \boxed{A_2 = 0.100 \text{ m}}$$

4. (20 points = 10 + 10)

The diagram below is a vector field plot for a mechanical system consisting of a particle that moves along a straight line (the x axis) under the action of a force that depends on its position x and its velocity v. In answering the following questions, feel free to draw on this diagram and label your drawings so you can refer to them in your answers.



a) Describe the nature of the force, and the resulting motion, as fully as you can.

1) Potential wells at $x = -a, a$;

2) Damped oscillation due to velocity-dependent damping force

3) mass eventually comes to rest at $x = a$ or $x = -a$.

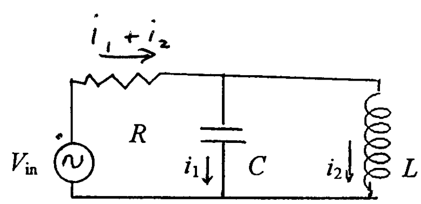
(continued)

4. (continued)

b) Under some circumstances the motion of this system can show extreme sensitivity to initial conditions. Discuss the nature of this sensitivity and how it can be predicted from the vector field plot.

Small changes in x_0 or v_0 may determine whether the mass comes to rest at $x = a$ or $x = -a$.

5. (35 points = 10 + 10 + 10 + 5)



a) Using the complex number representation with the circuit shown, obtain a sufficient number of equations so that the equations can be solved to find the current in each branch of the circuit, in terms of R, L, C, V_{in}, and ω. Do not solve the equations.

$$V_{in} - (i_1 + i_2) R - i_1 \frac{1}{j\omega C} = 0$$

$$V_{in} - (i_1 + i_2) R - i_2 \, j\omega L = 0$$

$$+ i_1 \frac{1}{j\omega C} - i_2 \, j\omega L = 0$$

(any two of these)

b) There is one frequency for which the current in the resistor R is zero. Explain how this is possible, and derive an expression for this frequency, in terms of the quantities named above. Hint: It is not necessary to solve the circuit equations you found in part (a).

i_1 and i_2 are always $\frac{1}{2}$ cycle out of phase. If they are also equal in magnitude, then $i_1 + i_2 = 0$ or $i_1 = -i_2$.

For any ω, $i_1 \frac{1}{j\omega C} = i_2 \, j\omega L$

$$i_1 = (j\omega L)(j\omega C)\, i_2 = -\omega^2 LC \, i_2$$

If $i_1 + i_2 = 0$, $\omega^2 LC = 1$, $\boxed{\omega = \frac{1}{\sqrt{LC}}}$

(continued)

5 (continued)

c) Suppose $R = 200\ \Omega$, $L = 0.50$ H, $C = 8.0\ \mu$F, and $V_{in} = 75$ V. Obtain a numerical value for the angular frequency for which the current in R is zero, and find the current in L at this frequency.

$$\omega = \frac{1}{\sqrt{(0.5\,H)(8.0\times 10^{-6}\,F)}} = \frac{1}{2}\times 10^{3}\ s^{-1} = \boxed{500\ s^{-1}}$$

$$V_L = V_{in}, \qquad i_2 = \frac{V_L}{j\omega L} = \frac{75\,V}{j(500\,s^{-1})(0.5\,H)}$$

$$i_2 = -j\,(0.30\ A)$$

(magnitude 0.30 A, lags V_{in} by $90°$)

d) When the angular frequency has the value found in (c), what is the *phase* of the current in L, relative to the input voltage? Explain your answer briefly.

i_2 lags V_{in} by $90°$ because of factor $-j$

6. (30 points = 7 + 8 + 7 + 8)

A nylon rope, used for mountain climbing, is stretched out horizontally, with a tension of 250 N, and the end at $x = 0$ is given a sinusoidal transverse motion. The other end, at $x = 50$ m, is tied to a device that maintains tension but absorbs the arriving wave, so that there is no reflected wave. The wave function is

$$\psi(x,t) = (0.050\,\text{m})\cos\left[(30\,\text{s}^{-1})t - (0.60\,\text{m}^{-1})x\right].$$

a) Determine the wave speed.

$\omega = 30\ s^{-1}$ $\quad k = 0.60\ m^{-1}$ $\quad \omega = ck$

$$c = \frac{\omega}{k} = \frac{30\ s^{-1}}{0.6\ m^{-1}} = 50\ m/s$$

b) Determine the maximum transverse speed of a point on the rope.

$$v_y = \frac{\partial \psi}{\partial t} = (0.05\ m)(30\ s^{-1})(-1)\sin\left[(30\ s^{-1})t - (0.60\ m^{-1})x\right]$$

$$v_y\Big|_{max} = (0.05\ m)(30\ s^{-1}) = 1.5\ m/s$$

(continued)

6. (continued)

c) Determine the mass per unit length of the rope.

$$c = \sqrt{\frac{F}{\mu}} \Rightarrow \mu = \frac{F}{c^2}$$

$$\mu = \frac{250\ N}{(50\ m/s)^2} = \cancel{0.0}\ 0.10\ kg/m$$

d) Determine the maximum transverse force exerted *by* the part of the rope to the left of a given point, *on* the part of the rope to the right of that point.

$$F_y = F\frac{\partial \psi}{\partial x}$$

$$= (250 N)(0.050 m)(0.6 m^{-1})\cancel{\ }\sin[\qquad]$$

$$F_y)_{max} = 7.5\ N$$

Useful Equations:

$$T=2\pi\sqrt{\frac{m}{k}}, \qquad f=\frac{1}{T}=\frac{1}{2\pi}\sqrt{\frac{k}{m}}, \qquad \omega_o=\sqrt{\frac{k}{m}}, \qquad \gamma=\frac{b}{2m}. \qquad (6), (7)*$$

$$m\ddot{x}+b\dot{x}+kx=0; \quad \ddot{x}+2\gamma\dot{x}+\omega_o{}^2x=0, \quad \omega_d=\sqrt{\omega_o{}^2-\gamma^2}, \quad \gamma_d=\sqrt{\gamma^2-\omega_o{}^2}. \quad (11)$$

$$x=e^{-\gamma t}\left(Ae^{-\gamma_d t}+Be^{\gamma_d t}\right), \quad A=-\frac{(\gamma-\gamma_d)x_o+v_o}{2\gamma_d}, \quad B=\frac{(\gamma+\gamma_d)x_o+v_o}{2\gamma_d}. \qquad (12),(13)$$

$$x=e^{-\gamma t}(A\cos\omega_d t+B\sin\omega_d t), \qquad A=x_o, \qquad B=\frac{v_o+\gamma x_o}{\omega_d}. \qquad (14),(15)$$

$$x=(A+Bt)e^{-\gamma t}, \qquad A=x_o, \qquad B=\gamma x_o+v_o. \qquad (16),(17)$$

$$\left|\frac{\Delta E}{E}\right| \cong \frac{4\pi\gamma}{\omega_d} \cong \frac{4\pi\gamma}{\omega_o}, \qquad \left|\frac{\Delta E_{\text{per radian}}}{E}\right| = \frac{2\gamma}{\omega_o}, \qquad Q=\frac{\omega_o}{2\gamma}, \qquad E=E_o e^{-2\gamma t}. \ (27)\text{-}(30)$$

$$F=F_o\cos\omega t, \qquad x=A'\cos(\omega t+\varphi), \qquad A'=\frac{F_o/m}{\omega_o{}^2-\omega^2}, \qquad (33)$$

$$A'=\frac{F_o/m}{\sqrt{\left(\omega^2-\omega_o{}^2\right)^2+(2\gamma\omega)^2}}, \quad \tan\varphi=\frac{2\gamma\omega}{\omega^2-\omega_o{}^2}, \quad A'\cong\frac{F_o/2m\omega_o}{\sqrt{(\omega_o-\omega)^2+\gamma^2}}. \quad (37),(41)$$

$$x=e^{-\gamma t}\left(A\cos\omega_d t+B\sin\omega_d t\right) + A'\cos(\omega t+\varphi). \qquad (45)$$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots.$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots, \qquad \cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots.$$

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^2 x\,dx=\frac{1}{2\pi}\int_0^{2\pi}\cos^2 x\,dx=\frac{1}{2}.$$

$$\psi(x,t) = A\cos\left[2\pi\left(ft-\frac{x}{\lambda}\right)\right] = A\cos\left[2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right)\right] = A\cos\left[2\pi f\left(t-\frac{x}{c}\right)\right]$$

$$= A\cos(\omega t-kx) = A\cos\left[\omega\left(t-\frac{x}{c}\right)\right]. \qquad c\ (=v) = \sqrt{\frac{F}{\mu}}.$$

* Numbers refer to equations in the notes on "Simple Harmonic Motion."