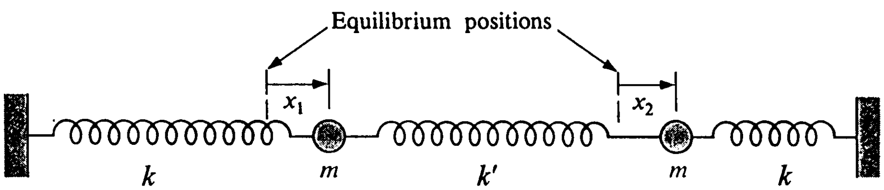
8 Coupled Oscillators and Normal Modes — Fall 2003

An undamped harmonic oscillator (a mass m and a Hooke's-law spring with force constant k) has only one natural frequency of oscillation, $\omega_o = \sqrt{k/m}$. But when two or more such oscillators interact, several natural frequencies are possible.

Let's consider a system of masses and Hooke's-law springs that has a stable equilibrium position, such that each mass can vibrate around its equilibrium position. Are there possible motions in which every mass moves with simple harmonic motion, all masses with the same frequency? Such a motion, when it exists, is called a *normal-mode* motion. We will now develop general methods for finding the possible normal modes of such a system and their associated frequencies. We'll assume throughout that the spring forces are *linear* functions of displacement. We'll illustrate the general method by use of the following specific example.

Example

Let's consider the system shown below. The two masses move along a straight line. In the equilibrium positions, the springs are neither stretched nor compressed, and the coordinates x_1 and x_2 are the displacements of the particles from equilibrium.



If $k' = 0$ (i.e., if the center spring is removed), we have two uncoupled harmonic oscillators; each one can vibrate with angular frequency $\omega = \sqrt{k/m}$, with arbitrary amplitude and phase. When the central spring is included, there are two cases where the masses can oscillate with the same frequency:

1) If $x_1 = x_2$ at each instant, then spring k' is never stretched or compressed, and it can be ignored. The two masses vibrate sinusoidally, in phase, with the same angular frequency $\omega = \sqrt{k/m}$, and with equal amplitudes.

2) If $x_1 = -x_2$ at each instant, the midpoint of spring k' is stationary, and the force it exerts on each mass is like that of a spring with force constant $2\,k'$. The *total* force on each mass is the same as for a spring with force constant $k + 2\,k'$. In this case, the two masses move sinusoidally with angular frequency $\omega = \sqrt{(k + 2k')/m}$, with equal amplitudes but a half-cycle out of phase.

Thus this system has two *normal modes,* one with angular frequency $\omega = \sqrt{k/m}$, the other with $\omega = \sqrt{(k + 2k')/m}$. Each mode has a characteristic vibration pattern (i.e., a relation between the amplitudes and phases of the motions of the two masses).

General Method

In our example, the symmetry of the problem allowed us to *guess* the normal modes, but we need a more systematic and general approach. To develop this approach, we start with the equations of motion, from $\Sigma F = ma$. They are

$$\begin{aligned} -kx_1 + k'(x_2 - x_1) &= m\ddot{x}_1, \\ -kx_2 - k'(x_2 - x_1) &= m\ddot{x}_2. \end{aligned} \tag{1}$$

(Be sure you understand the various $+$ and $-$ signs in these equations.)

Now we *guess* that these equations have a solution in the form

$$x_1 = a_1 \cos(\omega t + \varphi), \qquad x_2 = a_2 \cos(\omega t + \varphi), \tag{2}$$

where ω is not yet known and the amplitudes a_1 and a_2 may be related.
To test whether (or under what circumstances) these expressions really do satisfy Eqs. (1), we carry out the derivatives and substitute back into Eqs. (1). After dividing out the common factor $\cos(\omega t + \varphi)$ and re-arranging, we get

$$\begin{aligned} -ka_1 + k'(a_2 - a_1) &= -m\omega^2 a_1, \\ -ka_2 - k'(a_2 - a_1) &= -m\omega^2 a_2, \end{aligned}$$

or

$$\begin{aligned} (k + k' - m\omega^2)a_1 - k'a_2 &= 0, \\ -k'a_1 + (k + k' - m\omega^2)a_2 &= 0. \end{aligned} \tag{3}$$

That is, Eqs. (2) are a solution of the equations of motion, Eqs. (1), if (and *only* if) Eqs. (3) are satisfied.

Eqs. (3) are a pair of *simultaneous, homogeneous* equations for the amplitudes a_1 and a_2. They *always* have the trivial solution $a_1 = a_2 = 0$. A fundamental theorem of linear algebra states that *non-trivial* solutions for these equations exist if (and only if) the *determinant* of the system is zero. Thus a necessary and sufficient condition for the existence of non-trivial solutions is

$$\begin{vmatrix} k + k' - m\omega^2 & -k' \\ -k' & k + k' - m\omega^2 \end{vmatrix} = 0. \tag{4}$$

(For further discussion of this point, please refer to Section 9, pages 9-4 and 9-5.)

The values of k, k', and m are fixed, so we conclude that the assumed solution is valid only for certain special values of ω, the values that satisfy Eq. (4). This equation for ω is called the *secular equation*. When we multiply out the determinant, we get

$$(k + k' - mw^2)^2 = k'^2 \qquad \text{and} \qquad k + k' - mw^2 = \pm k'. \tag{5}$$

Assuming $\omega \geq 0$, we get

$$\omega = \sqrt{\frac{k}{m}} \qquad \text{and} \qquad \omega = \sqrt{\frac{k + 2k'}{m}}, \tag{6}$$

in agreement with our previous result.

Each value of ω in Eqs. (6) is a normal-mode frequency. To get the amplitude relations for each normal mode, we substitute each value for ω back into Eqs. (3):

$$\text{For } \omega^2 = \frac{k}{m}, \qquad \begin{aligned} k' a_1 - k' a_2 &= 0, \\ -k' a_1 + k' a_2 &= 0. \end{aligned} \qquad \text{I.e.,} \quad a_1 = a_2. \tag{7}$$

$$\text{For } \omega^2 = \frac{k + 2k'}{m}, \qquad \begin{aligned} -k' a_1 - k' a_2 &= 0, \\ -k' a_1 - k' a_2 &= 0. \end{aligned} \qquad \text{I.e.,} \quad a_1 = -a_2. \tag{8}$$

In each case, the equations don't give us specific values for a_1 and a_2, but they show that they must be related in a very particular way.

Each of these possibilities is a *normal mode*, with a definite frequency and a relation between the amplitudes, describing the pattern of the motion. The most general motion of the system is a superposition of these normal modes, with arbitrary phases. Call the modes Mode 1 and Mode 2, with

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \text{and} \qquad \omega_2 = \sqrt{\frac{k + 2k'}{m}}. \tag{9}$$

Then for Mode 1, with $a_1 = a_2 = a$,

$$\begin{aligned} x_1 &= a \cos\left(\omega_1 t + \varphi_1\right), \\ x_2 &= a \cos\left(\omega_1 t + \varphi_1\right), \end{aligned} \tag{10}$$

and for Mode 2, with $a_1 = -a_2 = b$,

$$\begin{aligned} x_1 &= b \cos\left(\omega_2 t + \varphi_2\right), \\ x_2 &= -b \cos\left(\omega_2 t + \varphi_2\right). \end{aligned} \tag{11}$$

The most general motion of the system is described by equations with four arbitrary constants a, b, φ_1, and φ_2, determined by the initial conditions:

$$\begin{aligned} x_1 &= a \cos\left(\omega_1 t + \varphi_1\right) + b \cos\left(\omega_2 t + \varphi_2\right), \\ x_2 &= a \cos\left(\omega_1 t + \varphi_1\right) - b \cos\left(\omega_2 t + \varphi_2\right). \end{aligned} \tag{12}$$

The amplitude factor a and the phase angle φ_1 are associated with Mode 1, and b and φ_2 are associated with Mode 2.

Normal Coordinates

Suppose we call the two sinusoidal variables q_1 and q_2:

$$q_1 = a\cos(\omega_1 t + \varphi_1), \qquad \text{Then} \qquad x_1 = q_1 + q_2,$$
$$q_2 = b\cos(\omega_2 t + \varphi_2). \qquad\qquad x_2 = q_1 - q_2. \tag{13}$$

We can express the equations of motion, Eqs. (1), in terms of the variables q_1 and q_2. Substituting Eqs. (13) into Eqs. (1) and re-arranging, we get

$$m(\ddot{q}_1 + \ddot{q}_2) = -kq_1 - (k + 2k')q_2,$$
$$m(\ddot{q}_1 - \ddot{q}_2) = -kq_1 + (k + 2k')q_2. \tag{14}$$

Now note that by simply adding or subtracting these equations, we can obtain separate equations for q_1 and q_2:

$$m\ddot{q}_1 = -kq_1,$$
$$m\ddot{q}_2 = -(k + 2k')q_2. \tag{15}$$

These equations show that the coordinate q_1 varies sinusoidally with angular frequency $\omega_1 = \sqrt{k/m}$, and q_2 with angular frequency $\omega_2 = \sqrt{(k + 2k')/m}$, as we should expect. Thus if we had been lucky enough (or clever enough) to use q_1 and q_2 as coordinates at the start, we would have obtained the normal modes immediately. Each normal mode consists of a motion in which only one of the q's is different from zero.

The q's are called *normal coordinates* for this system, and the transformation from the x's to the q's, given by Eqs. (13), is a *normal-coordinate transformation.* You can verify that the inverse transformation, giving the q's in terms of the x's, is

$$q_1 = \tfrac{1}{2}(x_1 + x_2),$$
$$q_2 = \tfrac{1}{2}(x_1 - x_2). \tag{16}$$

Energy

The total energy of the system can be expressed in terms of the normal coordinates. In terms of the original coordinates x_1 and x_2, the total energy is

$$E = T + V = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 + \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}kx_2^2 + \tfrac{1}{2}k'(x_1 - x_2)^2. \tag{17}$$

Substituting Eqs. (13) into this expression and re-arranging, we get

$$E = m(\dot{q}_1^2 + \dot{q}_2^2) + kq_1^2 + (k + 2k')q_2^2$$
$$= \left[m\dot{q}_1^2 + kq_1^2\right] + \left[m\dot{q}_2^2 + (k + 2k')q_2^2\right]. \tag{18}$$

We see that the total energy separates into terms containing only q_1 and its derivative and terms containing only q_2 and its derivative. We invite you to verify that a similar separation occurs with the Lagrangian function $L = T - V$.

Limitations

Our entire analysis has made use of the fact that the restoring forces are *linear* functions of the coordinates. When non-linear forces are present, in general there is no such thing as normal-mode motion. Just as a single-mass oscillator with a non-linear restoring force has a frequency that depends on the amplitude of the motion, so it is with more complex systems when non-linear forces are present.

Matrix Formulation

The normal-coordinate transformation given by Eqs. (13) and (16) can be expressed compactly using matrix language. Each set of coordinates is represented by a one-column matrix, and the linear transformation from one to the other is given by a square matrix **A**. In the above example,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{19}$$

Then the transformation given by Eqs. (13) can be written simply as $\mathbf{x} = \mathbf{A}\mathbf{q}.$

Similarly, the inverse transformation can be written as $\mathbf{q} = \mathbf{A}^{-1}\mathbf{x},$ where

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

You can verify that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix (or unit matrix):

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Matrix methods are a very powerful tool for analyzing normal modes of complex systems. We'll discuss them in a separate chapter.

(If you need to review matrix multiplication or other aspects of matrix algebra, please consult Edwards and Penney *Differential Equations and Boundary Value Problems*, 2nd ed., Section 5.1, pp. 284-290. This section also includes a brief discussion of determinants.)