

How to efficiently deal with very large and very small numbers

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There exists a highly convenient and elegant way in mathematics to deal with numbers which are so big or so small that the conventional way of writing them is *extremely* tedious. This is the powers-of-ten exponential notation. In these notes I briefly describe how this notation is handled. I expect you to be able to use it.

I. INTRODUCTION

The times are over when dealing with numbers only meant dealing with numbers between 1 and 100. There are countless examples, surely not just in physics, where very big or very small numbers come up, but writing them down in the conventional way, even though of course possible, is very tedious. Here are some (hopefully convincing) examples:

- Modern computer hard disks can store something like 1000000000000 bytes of data.
- The 2010 US Gross National Debt amounts to approximately \$1372400000000.
- The mass of the sun, to one significant digit, is approximately 20000000000000000000000000000000 kg.
- The probability of winning the SuperEnalotto in Italy (betting 6 numbers out of 90 correctly) is approximately 0.00000000161.
- the clock cycle in current computers is on the order of 0.000000004 s.

These numbers are correct, but horribly written down. Notice how you end up counting zeros, or places, or how far the decimal point moved? Rather than doing this counting, which we somehow need to do anyways, we could use a notation which saves us the tedium of counting by simply *telling* us how many zeros or how many places to shift the decimal point. This notation exists, and using it, the above statements get rephrased like this:

- Hard disks: 10^{12} bytes of data.
- 2010 US Gross National Debt: $\$ 1.3724 \times 10^{13}$.
- Mass of the sun: 2×10^{30} kg.
- Probability of winning the SuperEnalotto: 1.61×10^{-9} .
- Clock cycle in current computers: 4×10^{-10} s.

Especially once we do not just want to write down these numbers but actually do something with them (such as a bit of math), the latter notation is infinitely more useful. Besides, it's not that hard to get used to it.

II. BASIC IDEA

The basic idea is to use powers of ten to denote numbers. This is convenient because powers of ten very quickly can get very large or very small—which is exactly what we need now. And we use powers of 10 rather than any other number because our number system is based on the base 10. The latter is most likely because we have 10 fingers, and all math seems to have started with counting with your hands. Mathematically, though, base 10 is not very convenient. Base 12 would often be much nicer (because 12 has more dividers than 10), but it's a safe bet that this is never going to happen. (But recall: 12 inches to the foot, and 2×12 hours to the day, 5×12 seconds to the minute, ...)

First notice this:

$$\begin{aligned} 10 &= 10^1 \\ 100 &= 10^2 \\ 1000 &= 10^3 \\ 10000 &= 10^4 \\ 100000 &= 10^5 . \end{aligned}$$

As you see, the *exponent* at 10 just tells you, how many zeros you need to put *right* behind a “1”.

Next, if you have numbers smaller than 1, it works like this:

$$\begin{aligned} 0.1 &= 10^{-1} \\ 0.01 &= 10^{-2} \\ 0.001 &= 10^{-3} \\ 0.0001 &= 10^{-4} \\ 0.00001 &= 10^{-5} . \end{aligned}$$

Now the number in the exponent is negative, and it tells you by how many places you have to move the decimal point to the *right* in order to get “1”, or how many times you have to divide 1 by 10.

For consistency, we will also have

$$1 = 10^0 .$$

If you want numbers that are not simply powers of ten, you create them by multiplying a suitable power of ten by a prefactor. For instance:

$$\begin{aligned} 20 &= 2 \times 10 \\ 2000 &= 2 \times 1000 = 2 \times 10^3 \\ 56000000 &= 56 \times 1000000 = 56 \times 10^6 = 5.6 \times 10^7 \\ 0.000037 &= 3.7 \times 0.00001 = 3.7 \times 10^{-5} . \end{aligned}$$

III. SOME SIMPLE LAWS FOR POWERS OF TEN

Here are some simple laws for powers of ten. All of them in fact hold for powers of any number, but since 10 is right now the only one we need, we keep it specific. The most important thing is that *multiplying* powers of 10 implies *adding* the corresponding exponents:

$$\begin{aligned}10^3 \times 10^5 &= 10^8 \\10^4 \times 10^{-3} &= 10^1 \\10^7 \times 10^{-11} &= 10^{-4} \\10^5 \times 10^{-5} &= 10^0 = 1.\end{aligned}$$

Next, as we have already seen with positive and negative exponents, the negative exponents seem to make the numbers very small. In fact, check this out:

$$\frac{1}{10\,000} = 0.0001 = 10^{-4}. \quad (1)$$

This in fact means that for any number n we have

$$10^{-n} = \frac{1}{10^n}.$$

The following examples illustrate this:

$$\begin{aligned}10^{-3} &= \frac{1}{10^3} \\ \frac{1}{10^{-7}} &= \frac{1}{\frac{1}{10^7}} = 10^7.\end{aligned}$$

Make sure you understand also the second of these examples.

So we can now look at things such as

$$\frac{10^3 \times 10^2 \times 10^{-5}}{10^7 \times 10^{-1}} = 10^{3+2-5-7+1} = 10^{-6}.$$

What if we now have powers of ten and some other prefactors, such as in numbers like 3.2×10^5 ? What if we now need to say, multiply such numbers? The answer is this: Since it doesn't matter in which order we multiply numbers, we collect all the prefactors and deal with them separately, and also collect all the powers of 10 and deal with them separately as well. For instance, when we need to multiply the numbers 3.2×10^5 , 5×10^{-2} and 3.75×10^{-4} , we do it like this:

$$\begin{aligned}(3.2 \times 10^5) \times (5 \times 10^{-2}) \times (3.75 \times 10^{-4}) \\&= 3.2 \times 10^5 \times 5 \times 10^{-2} \times 3.75 \times 10^{-4} \\&= (3.2 \times 5 \times 3.75) \times (10^5 \times 10^{-2} \times 10^{-4}) \\&= 60 \times 10^{5-2-4} \\&= 60 \times 10^{-1} \\&= 6.\end{aligned}$$

IV. PREFIXES

We have also learned that prefixes exist for certain powers of ten. For instance, “kilo” means 10^3 , or “nano” means 10^{-9} . This is *literally* true, because we can simply rewrite $2.3 \text{ nm} = 2.3 \times 10^{-9} \text{ m}$. This becomes important if we do calculations and we end up with various instances of, say, length, but they have different prefixes. In this case it's a good idea to get everything to the *same* unit (for instance, everything in meter or everything in nanometer), because we might find instances where we could cancel such units. You cannot right away cancel meter against nanometer, even though both are lengths. But once you replace the “nano” by its numerical value 10^{-9} , you have two instances of meters, and those you can cancel (if they are upstairs and downstairs in a fraction). For instance:

$$\frac{3.2 \text{ m}}{1.6 \text{ nm}} = \frac{3.2 \text{ m}}{1.6 \times 10^{-9} \text{ m}} = \frac{3.2}{1.6} \times 10^9 = 2 \times 10^9.$$

Notice that if you had left out the units and merely wrote down $\frac{3.2}{1.6}$, you would have gotten the answer 2, *which is a factor of one billion wrong!* You might not deeply care about the homework problems we give you, but if ever in real life you happen to have to do some calculation of that type for your boss, and you then get the answer wrong by a factor of one billion, you will lose your job more quickly than you can blink your eyes (which, incidentally, takes a few deciseconds...).

Here's a real life example: What's the frequency of UV light? Say we know that the wavelength is 300 nm, and we also know that the speed of light is $3 \times 10^8 \frac{\text{m}}{\text{s}}$. From the wave equation $c = \lambda f$ we find

$$\begin{aligned}f &= \frac{c}{\lambda} = \frac{3 \times 10^8 \frac{\text{m}}{\text{s}}}{300 \text{ nm}} \quad (\text{rewrite nm as powers of ten times meter}) \\&= \frac{3 \times 10^8 \frac{\text{m}}{\text{s}}}{300 \times 10^{-9} \text{ m}} \quad (\text{shift the exponent in the denominator}) \\&= \frac{3 \times 10^8 \frac{\text{m}}{\text{s}}}{3 \times 10^{-7} \text{ m}} \quad (\text{separate prefactor, powers of ten, and units}) \\&= \frac{3}{3} \times 10^{8+7} \times \frac{\text{m}}{\text{m}} \quad (\text{simplify and also cancel units}) \\&= 1 \times 10^{15} \frac{1}{\text{s}} \quad (\text{rename the unit as it's appropriate here}) \\&= 10^{15} \text{ Hz} . \quad (\text{done!})\end{aligned}$$

V. CONCLUSION

It is not terribly difficult to handle the power-of-ten notation. Once you have practiced it a little bit, it will save you a lot of time. It is much less error-prone than writing the clumsy numbers we started this discussion with and then count zeros or places. It goes along with very simple rules for how to manipulate the exponents once you need to do a calculation.

And, if nothing else, for the context of the present course, the probably best motivation for learning it is this: I expect you to be able to handle this!