ME 24-731
Conduction and Radiation Heat Transfer

Solution to Mid-Term Examination
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Total = 25 points

1. (a) The Biot number hd/k for the cube $0.01 \ll 1$. So a lumped capacitance analysis is valid.
   (b) From a heat balance on the cude, we can write

$$\rho CV \frac{\partial T}{\partial t} = 2Aq'' + 2Ah\left(T_\infty - T\right)$$

   with initial conditions T=T_1=30°C. The solution to the problem is:

$$T(t) = T_\infty + \frac{q''}{h} + \left(T_1 - T_\infty - \frac{q''}{h}\right) exp\left(-\frac{2hA}{\rho CV}t\right)$$

   Substituting values for properties and geometric parameters, we get

$$T(t) = 220 - 190.0 exp\left(-\frac{t}{1209.60}\right)$$

   (c) At steady state ($t \rightarrow \infty$), T=220°C from the above.

2. The governing equation in terms of $\theta = (T - T_a)/(T_i - T_a)$ is:

$$\frac{1}{\alpha}\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}$$

with boundary and initial conditions

$$\begin{aligned}
-k\frac{\partial \theta}{\partial x}(0,y,t) &= -h\theta(0,y,t) \\
-k\frac{\partial \theta}{\partial y}(x,0,t) &= -h\theta(x,0,t) \\
\theta(L,y,t) &= 0 \\
\theta(x,L,t) &= 0 \\
\theta(x,y,0) &= 1
\end{aligned}$$

Postulating that $\theta = X(x,t)Y(y,t)$ we get

$$\frac{1}{X}\left(X'' - \frac{1}{\alpha}\frac{\partial X}{\partial t}\right) + \frac{1}{Y}\left(Y'' - \frac{1}{\alpha}\frac{\partial Y}{\partial t}\right) = 0$$

Therefore each term above is only a function of t. Thus

$$\begin{aligned}
\frac{1}{X}\left(X'' - \frac{1}{\alpha}\frac{\partial X}{\partial t}\right) &= \lambda_1(t) \\
\frac{1}{Y}\left(Y'' - \frac{1}{\alpha}\frac{\partial Y}{\partial t}\right) &= \lambda_2(t) \\
\lambda_1 + \lambda_2 &= 0
\end{aligned}$$

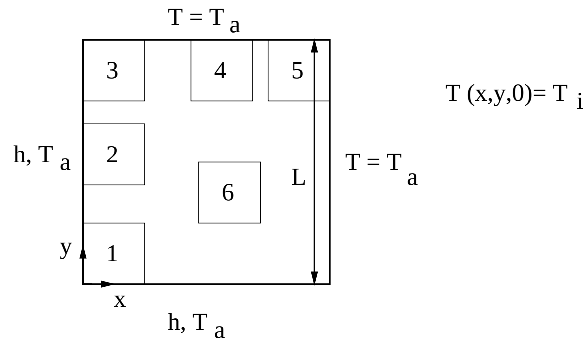

Figure 1: Geometry for Problems 2 and 3

From symmetry considerations, $\lambda_1 = \lambda_2$. Therefore, $\lambda_1 = \lambda_2 = 0$. For the $X(x,t)$ problem, we choose

$$\begin{aligned}
\frac{1}{\alpha}\frac{\partial X}{\partial t} &= X'' \\
X'(0,t) &= \frac{h}{k}X(0,t) \\
X(L,t) &= 0 \\
X(x,0) &= 1
\end{aligned}$$

The same set holds for the $Y(y,t)$ problem.
It is easiest to solve the X(x,t) problem by using a change of variables:

$$x^* = x - L$$

so that

$$\begin{aligned}
\frac{1}{\alpha}\frac{\partial X}{\partial t} &= X'' \\
X'(-L,t) &= \frac{h}{k}X(-L,t) \\
X(0,t) &= 0 \\
X(x^*,0) &= 1
\end{aligned}$$

Separating variables using $X(x^*,t) = \xi(x^*)\tau(t)$ we get

$$\begin{aligned}
\xi'' + \lambda^2\xi &= 0 \\
\tau' + \lambda^2\alpha\tau &= 0
\end{aligned}$$

where we have chosen a decaying exponential in time and sines and cosines in the x (homogeneous) direction. The solution therefore is

$$X(x^*,t) = \left(C_1\cos\lambda x^* + C_2\sin\lambda x^*\right)e^{-\lambda^2\alpha t}$$

Applying the $x^* = 0$ boundary condition we get $C_1 = 0$. Applying the $X^* = -L$ boundary condition, we get

$$\tan\lambda_n L = -\frac{k}{h}\lambda_n \quad n = 0,1,2,\ldots$$

The eigenvalues λ_n are solutions to the above transcendental equation. Thus, the total solution is:

$$X(x^*,t) = \sum_{n=0}^{\infty} D_n \sin \lambda_n x^* e^{-\lambda_n^2 \alpha t}$$

Applying the initial condition gives

$$1 = \sum_{n=0}^{\infty} D_n \sin \lambda_n x^*$$

We can now exploit the orthogonality of $\sin \lambda_n x^*$ to obtain D_n as

$$D_n = \frac{\int_{-L}^{0} \sin \lambda_n x^* dx^*}{\int_{-L}^{0} \sin^* \lambda_n x^* dx^*}$$

3. For each control volume, the discrete equation is of the form:

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + b$$

   (a) For control volume 1:

$$a_E = \frac{k\Delta y}{\Delta x};\ \ a_W = 0;\ \ a_N = \frac{k\Delta x}{\Delta y};\ \ a_S = 0;\ \ a_{BW} = \frac{hk\Delta y}{0.5\Delta x h + k};\ \ a_{BS} = \frac{hk\Delta x}{0.5\Delta y h + k};\ \ a_P^0 = \frac{\rho C \Delta x \Delta y}{\Delta t}$$

$$a_P = a_E + a_N + a_{BW} + a_{BS} + a_P^0\ \ ; b = (a_{BW} + a_{BS})T_a + a_P^0 T_P^0$$

   (b) For control volume 2:

$$a_E = \frac{k\Delta y}{\Delta x};\ \ a_W = 0;\ \ a_N = \frac{k\Delta x}{\Delta y};\ \ a_S = \frac{k\Delta x}{\Delta y};\ \ a_{BW} = \frac{hk\Delta y}{0.5\Delta x h + k};\ \ a_P^0 = \frac{\rho C \Delta x \Delta y}{\Delta t}$$

$$a_P = a_E + a_N + a_S + a_{BW} + a_P^0\ \ ;\ \ b = a_{BW} T_a + a_P^0 T_P^0$$

   (c) For control volume 3:

$$a_E = \frac{k\Delta y}{\Delta x};\ \ a_W = 0;\ \ a_N = 0;\ \ a_S = \frac{k\Delta x}{\Delta y};\ \ a_{BW} = \frac{hk\Delta y}{0.5\Delta x h + k};\ \ a_{BN} = \frac{k\Delta x}{0.5\Delta y};\ \ a_P^0 = \frac{\rho C \Delta x \Delta y}{\Delta t}$$

$$a_P = a_E + a_N + a_S + a_{BW} + a_{BN} + a_P^0;\ \ b = (a_{BW} + a_{BN})T_a + a_P^0 T_P^0$$

   (d) For control volume 4:

$$a_E = \frac{k\Delta y}{\Delta x};\ \ a_W = \frac{k\Delta y}{\Delta x};\ \ a_N = 0;\ \ a_S = \frac{k\Delta x}{\Delta y};\ \ a_{BN} = \frac{k\Delta x}{0.5\Delta y};\ \ a_P^0 = \frac{\rho C \Delta x \Delta y}{\Delta t}$$

$$a_P = a_E + a_W + a_S + a_{BN} + a_P^0;\ \ b = a_{BN} T_a + a_P^0 T_P^0$$

   (e) For control volume 5:

$$a_E = 0;\ \ a_W = \frac{k\Delta y}{\Delta x};\ \ a_N = 0;\ \ a_S = \frac{k\Delta x}{\Delta y};\ \ a_{BE} \frac{k\Delta y}{0.5\Delta x};\ \ a_{BN} = \frac{k\Delta x}{0.5\Delta y};\ \ a_P^0 = \frac{\rho C \Delta x \Delta y}{\Delta t}$$

$$a_P = a_W + a_S + a_{BE} + a_{BN} + a_P^0;\ \ b = (a_{BE} + a_{BN})T_a + a_P^0 T_P^0$$

   (f) For control volume 6:

$$a_E = \frac{k\Delta y}{\Delta x};\ \ a_W = \frac{k\Delta y}{\Delta x};\ \ a_N = \frac{k\Delta x}{\Delta y};\ \ a_S = \frac{k\Delta x}{\Delta y};\ \ a_P^0 = \frac{\rho C \Delta x \Delta y}{\Delta t}$$

$$a_P = a_E + a_W + a_N + a_S + a_P^0;\ \ b = a_P^0 T_P^0$$