ME 24-731
Conduction and Radiation Heat Transfer

Solution to Assignment No: 5
Due Date: March 21, 2000
Spring 2000
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1. The problem is split into two problems $T(x,y,t) = T_a(x,y) + T_b(x,y,t)$. The two problems satisfy:

$$\frac{\partial^2 T_a}{\partial x^2} + \frac{\partial^2 T_a}{\partial y^2} = 0$$

$$T_a(0,y) = T_a(x,0) = T_a(L,y) = 0; \;\; T_a(x,L) = T_1$$

$$\frac{1}{\alpha}\frac{\partial T_b}{\partial t} = \frac{\partial^2 T_b}{\partial x^2} + \frac{\partial^2 T_b}{\partial y^2}$$

$$T_b(0,y,t) = T_b(x,0,t) = T_b(L,y,t) = T_b(x,L,t) = 0$$

$$T_b(x,y,0) = T_i - T_a(x,y)$$

The solution to the T_a problem was done in class and is given by:

$$T_a(x,y) = \frac{2T_1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\frac{n\pi x}{L}\frac{\sinh\frac{n\pi y}{L}}{\sinh(n\pi)}$$

The solution to the T_b problem is obtained using separation of variables:

$$T_b(x,y,t) = X(x)Y(y)\mathscr{T}(t)$$

Separating variables and choosing harmonic functions in the homogeneous directions (x and y) we get

$$\begin{aligned} X(x) &= C_1\cos\lambda x + C_2\sin\lambda x \\ Y(y) &= C_3\cos\mu y + C_4\sin\mu y \\ \mathscr{T}(t) &= C_5 exp\left(-\alpha\left(\lambda^2+\mu^2\right)t\right) \end{aligned}$$

Applying boundary conditions in the x direction we get

$$X(x) = C_2\sin\lambda_n x; \quad \lambda_n = \frac{n\pi}{L} \;\; n = 1,2,3...$$

Similarly, in the y direction:

$$Y(y) = C_4\sin\mu_m y; \quad \mu_m = \frac{m\pi}{L} \;\; m = 1,2,3...$$

so that the total solution before applying the initial condition is

$$T_b(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin(\lambda_n x)\sin(\mu_m y)exp\left(-\alpha\left(\lambda_n^2+\mu_m^2\right)t\right)$$

Applying the initial condition

$$T_i - T_a(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin(\lambda_n x)\sin(\mu_m y)$$

Exploiting the orthogonality of $\sin(\lambda_n x)$, we can find C_{nm} as:

$$C_{nm} = \frac{\int_0^L\int_0^L (T_i - T_a(x,y))\sin(\lambda_n x)\sin(\lambda_m y)dxdy}{\int_0^L\int_0^L \sin^2(\lambda_n x)\sin^2(\lambda_m y)dxdy}$$

2. Assuming a lumped capacitance analysis is valid, the solution to the problem with a constant T_∞ is given by:

$$\theta = 1 - exp\left(-\frac{hA_s}{\rho CV}t\right)$$

where $\theta = (T-T_i)/(T_\infty - T_i)$ and A_s is the surface area of the body. Using Duhamel's superposition theorem, the solution to problem with $T_\infty = T_i + At$ is given by

$$T(t) - T_i = \frac{hA_s}{\rho CV}\int_0^t A\tau exp\left(-\frac{hA_s}{\rho CV}(t-\tau)\right)d\tau$$

3. Assume a uniform mesh with size $(\Delta x, \Delta y)$. The discrete equation for each of the points 1,2,3 and 4 in Figure 1 is given below.

   (a) Point 1:

$$a_E = \frac{k_e\Delta y}{\Delta x};\ a_W = 0.0;\ a_N = \frac{k_n\Delta x}{\Delta y};\ a_S = \frac{k_s\Delta x}{\Delta y};\ a_B = \frac{2k_P\Delta y}{\Delta x}$$

$$a_P = a_E + a_N + a_S + a_B + \frac{\rho C\Delta x\Delta y}{\Delta t};\ b = a_B T_b + \frac{\rho C\Delta x\Delta y}{\Delta t}T_P^0$$

   where T_b is the boundary temperature and T_P^0 is the temperature at the previous time step at the point P.

   (b) Point 2:

$$a_E = \frac{k_e\Delta y}{\Delta x};\ a_W = 0.0;\ a_N = 0.0;\ a_S = \frac{k_s\Delta x}{\Delta y};\ a_{Bw} = \frac{2k_P\Delta y}{\Delta x};\ a_{Bn} = \frac{2k_P\Delta x}{\Delta y};$$

$$a_P = a_E + a_S + a_{Bw} + a_{Bn} + \frac{\rho C\Delta x\Delta y}{\Delta t};\ b = a_{Bw}T_{bw} + a_{Bn}T_{bn} + \frac{\rho C\Delta x\Delta y}{\Delta t}T_P^0$$

   where T_{bw} and T_{bn} are the west and north boundary temperatures.

   (c) Point 3:

$$a_E = \frac{k_e\Delta y}{\Delta x};\ a_W = \frac{k_w\Delta y}{\Delta x};\ a_N = \frac{k_n\Delta x}{\Delta y};\ a_S = \frac{k_s\Delta x}{\Delta y}$$

$$a_P = a_E + a_W + a_N + a_S + \frac{\rho C\Delta x\Delta y}{\Delta t};\ b = \frac{\rho C\Delta x\Delta y}{\Delta t}T_P^0$$

(d) Point 4:

$$a_E = \frac{k_e \Delta y}{\Delta x};\;\; a_W = \frac{k_w \Delta y}{\Delta x};\;\; a_N = \frac{k_n \Delta x}{\Delta y};\;\; a_S = \frac{k_s \Delta x}{\Delta y}$$

$$a_P = a_E + a_W + a_N + a_S - S_P \Delta x \Delta y + \frac{\rho C \Delta x \Delta y}{\Delta t};\;\; b = S_C \Delta x \Delta y + \frac{\rho C \Delta x \Delta y}{\Delta t} T_P^0;\;\; S_P = -20;\;\; S_C = 1000$$

Face values of the conductivity are evaluated using harmonic-mean averaging:

$$k_e = \frac{2 k_P k_E}{k_P + k_E}$$

where k_P and k_E are evaluated using T_P and T_E respectively. At boundaries, we may either use k evaluated at the boundary temperature, or use the interior value k_P.

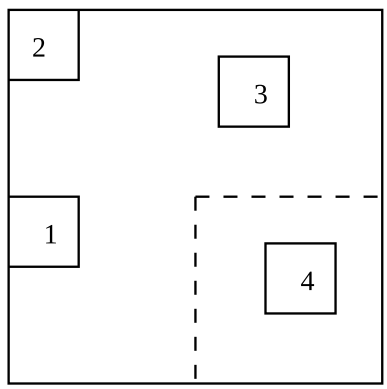

Figure 1: Grid for Problem 3