ME 24-731
Conduction and Radiation Heat Transfer

Solution to Assignment No: 3
Due Date: February 22, 2000
Spring 2000
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1. Define $\theta = T - T_1$ and split it $\theta = \theta_a + \theta_b$. θ_a satisfies

$$
\begin{aligned}
\frac{\partial^2 \theta_a}{\partial x^2} + \frac{\partial^2 \theta_a}{\partial y^2} &= 0 \\
\theta_a(x,0) &= 0 \\
-k\frac{\partial \theta_a}{\partial y}(x,L) &= q_0'' \\
-k\frac{\partial \theta_a}{\partial y}(0,y) &= 0 \\
-k\frac{\partial \theta_a}{\partial x}(L,y) &= h\theta_a(L,y)
\end{aligned}
$$

and θ_b satisfies

$$
\begin{aligned}
\frac{\partial^2 \theta_b}{\partial x^2} + \frac{\partial^2 \theta_b}{\partial y^2} &= 0 \\
\theta_b(x,0) &= 0 \\
-k\frac{\partial \theta_b}{\partial y}(x,L) &= 0 \\
-k\frac{\partial \theta_b}{\partial y}(0,y) &= 0 \\
-k\frac{\partial \theta_b}{\partial x}(L,y) &= h(\theta_b(L,y) - \theta_\infty)
\end{aligned}
$$

Each problem is solvable using separation of variables.

2. Using $U = \int_{T_0}^{T} \frac{k(T')}{k_0} dT'$, the conduction equation may be tranformed to give:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

The Dirichlet boundary condition $T = 0$ can be transformed to give:

$$U_b = \int_{T_0}^{0} \frac{k_0(1+\beta T')}{k_0} dT' = -T_0 - \frac{1}{2}\beta T_0^2$$

The Neumann condition at $y = L$ can be transformed to give

$$-k_0 \frac{\partial U}{\partial y}(x,L) = q_0''$$

We now have 4 inhomogeneous boundary conditions – 3 given-U_b boundary conditions, and one give-flux condition. By defining $\theta = U - U_b$ we can transform this problem into one with a single inhomogeneous given-flux boundary condition, and solve it using separation of variables. Once we have a $U(x,y)$ solution, we can get back T by solving the quadratic equation

$$U = (T - T_0) + \frac{1}{2}\beta\left(T^2 - T_0^2\right)$$

3. Can solve the problem in a $\theta_0 = \pi/4$ domain due to symmetry about $\theta = \pi/4$. For this domain, the governing equation and boundary conditions are:

$$\begin{aligned}
\frac{\partial^2 T}{r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r}\frac{\partial^2 T}{\partial \theta^2} &= 0 \\
T(r,0) &= 0 \\
T(r_1,\theta) = 0 & \\
\frac{\partial T}{\partial \theta}(r,\theta_0) &= 0 \\
-k\frac{\partial T}{\partial r}(r_2,\theta) &= q_0^{"}
\end{aligned}$$

Using separation of variables and choosing a periodic solution in the θ direction, we have

$$\begin{aligned}
T(r,\theta) &= R(r)\Theta(\theta) \\
\Theta &= C_1\cos(\lambda\theta) + C_2\sin(\lambda\theta) \\
R &= C_3 r^{\lambda} + C_4 r^{-\lambda}
\end{aligned}$$

Further, since $\Theta(r,0) = 0$, $C_1 = 0$. Also

$$\begin{aligned}
\frac{\partial \Theta}{\partial \theta}(r,\theta_0) &= 0 \\
\text{so that } \lambda_n &= \frac{n\pi}{2\theta} \quad n = 1,3,5...
\end{aligned}$$

Applying the condition at $r = r_1$, we get $C_4 = -C_3 r^{2\lambda_n}$. Collecting terms and consolidating constants, the total solution at this point is

$$T(r,\theta) = \sum_{n=1,3,5...}^{\infty} D_n \left(\left(\frac{r}{r_1}\right)^{\lambda_n} - \left(\frac{r}{r_1}\right)^{-\lambda_n} \right) \sin\left(\frac{n\pi}{2\theta_0}\theta\right)$$

Applying the last boundary condition,

$$\frac{q_0^{"}}{k} = -\sum_{n=1,3,5...}^{\infty} E_n \sin\left(\frac{n\pi}{2\theta_0}\theta\right)$$

where E_n is

$$E_n = D_n\lambda_n \left(\frac{r_2^{\lambda_n - 1}}{r_1^{\lambda_n}} + \frac{r_2^{-\lambda_n - 1}}{r_1^{-\lambda_n}} \right)$$

Thus, E_n can be found by invoking orthogonality:

$$E_n = \frac{\int_0^{\theta_0} q_0^{"} \sin(\lambda_n\theta)d\theta}{\int_0^{\theta_0} \sin^2(\lambda_n\theta)d\theta}$$