ME 24-731
Conduction and Radiation Heat Transfer

Solution to Assignment No: 2
Due Date: February 16, 2000
Spring 2000
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1. Writing a balance over an infinitesimal control volume:

$$\frac{dq_x}{dx} = h\left(T_\infty - T\right)P$$

Assuming a depth t into the page, we may write

$$q_x = -kA\frac{dT}{dx}; \quad A = bxt/L; \quad P = 2t$$

Further, defining $\theta = T - T_\infty$, we get

$$\frac{d}{dx}\left(x\frac{d\theta}{dx}\right) - m^2\theta = 0$$

where $m^2 = 2hL/kb$.

Using the transformation $z = 2mx^{\frac{1}{2}}$, we may write

$$\begin{aligned} \frac{d\theta}{dx} &= \frac{d\theta}{dz}mx^{-\frac{1}{2}} \\ \frac{d}{dx}\left(x\frac{d\theta}{dx}\right) &= \frac{d}{dz}\left(z\frac{d\theta}{dz}\right)\frac{z}{4x} = \left(z^2\frac{d^2\theta}{dz^2} + z\frac{d\theta}{dz}\right)\left(\frac{1}{4x}\right) \end{aligned}$$

so that

$$z^2\frac{d^2\theta}{dz^2} + z\frac{d\theta}{dz} - z^2\theta = 0$$

2. (a) How many terms you need depends on the location. Very accurate answers can be obtained with less than 50 terms.

   (b) Not shown.

   (c) At any $x = x_0$, the variation in y is monotonic because each term in the series only depends on $sinh(n\pi y/L)$. This automatically implies that the minimum is at $y = 0$ and the maximumis at $y = W$. Since this is true at all x, there are no maxima or minima in the domain.

   There is a theorem called the Maximin theorem for harmonic functions (i.e. solutions to Laplace's equation) which has a much more general proof. You can find this is *Introduction to Partial Differential Equations with Applications* by Zachmanoglu and Thoe, pp. 194-196, Williams and Wilkins, 1976, or any other book on partial differential equations.

   (d)

$$-k\frac{\partial T}{\partial y}|_{y=W} = -k\left(T_2 - T_1\right)\frac{\partial \theta}{\partial y}|_{y=W} \quad = \quad -k\frac{600}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\left(n\pi\right)sin\left(n\pi x\right)\frac{cosh\left(2n\pi\right)}{sinh\left(2n\pi\right)}$$

3. In all problems, we first remove one inhomogeneity by defining $\theta = T - T_1$.

   (a) After applying the all four boundary conditions we have:

   $$f(x) = \sum_{n=1}^{\infty} D_n sin\left(\frac{n\pi x}{L}\right) sinh\left(\frac{n\pi W}{L}\right)$$

   so that

   $$D_n sinh\left(\frac{n\pi W}{L}\right) = \frac{\int_0^L f(x) sin\frac{n\pi x}{L} dx}{\int_0^L sin^2\frac{n\pi x}{L} dx}$$

   Separation of variables can thus be used for this problem.

   (b) Can break the problem up into

   $$\theta = \theta_a(x,y) + \theta_b(x,y)$$

   θ_a has boundary conditions

   $$\begin{aligned} \theta_a(0,y) &= \theta_a(L,y) = \theta_a(x,0) = 0 \\ \theta_a(x,W) &= \theta_2(x) \end{aligned}$$

   θ_b has boundary conditions

   $$\begin{aligned} \theta_b(0,y) &= \theta_b(L,y) = 0 \\ \theta_b(x,0) &= \theta_3(x) \\ \theta_b(x,W) &= 0 \end{aligned}$$

   This can clearly be solved using separation of variables.

   (c) After applying the bounary conditions at x=0, x=L and y=0, we have:

   $$\theta(x,y) = \sum_{n=1}^{\infty} D_n sin\left(\frac{n\pi x}{L}\right) sinh\left(\frac{n\pi y}{L}\right)$$

   The convective boundary condition can be written as:

   $$h\left(\theta_\infty - \theta(x,W)\right) = k\frac{\partial\theta}{\partial y}(x,W)$$

   so that

   $$\frac{h}{k}\theta_\infty - \frac{h}{k}\sum_{n=1}^{\infty} D_n sin\left(\frac{n\pi x}{L}\right) sinh\left(\frac{n\pi W}{L}\right) = \sum_{n=1}^{\infty}\left(D_n\frac{n\pi}{L}\right) sin\left(\frac{n\pi x}{L}\right) cosh\left(\frac{n\pi W}{L}\right)$$

   so that

   $$\frac{h}{k}\theta_\infty = \sum_{n=1}^{\infty} D_n sin\left(\frac{n\pi x}{L}\right)\left(\frac{h}{k}sinh\left(\frac{n\pi W}{L}\right) + \frac{n\pi}{L}cosh\left(\frac{n\pi W}{L}\right)\right)$$

   We see that we can solve for D_n by invoking the orthogonlity of $sin(n\pi x/L)$. So separation of variables can be used successfully.

   (d) After applying all four boundary conditions we have

   $$T_2 = \sum_{n=1}^{\infty} D_n sin\left(\frac{n\pi x}{L}\right) sinh\left(\frac{n\pi(ax+b)}{L}\right)$$

   We see that the extra $sinh(n\pi(ax+b)/L)$ term destroys our ability to invoke the orthogonality of $sin(n\pi x/L)$. So we cannot use separation of variables.