

Finite Difference Method for the Solution of Laplace Equation

Laplace Equation is a second order partial differential equation(PDE) that appears in many areas of science an engineering, such as electricity, fluid flow, and steady heat conduction. Solution of this equation, in a domain, requires the specification of certain conditions that the unknown function must satisfy at the boundary of the domain. When the function itself is specified on a part of the boundary, we call that part the Dirichlet boundary; when the normal derivative of the function is specified on a part of the boundary, we call that part the Neumann boundary.

As in the case of ordinary differential equation, the idea of finite-difference-method(FDM) is to discretize the PDE by replacing the partial derivatives with their approximations, that is, finite differences. We will illustrate the scheme with Laplace's equation in the following.

Let us divide a two-dimensional region into small regions with increments in the x and y directions given as Δx and Δy , as shown in Figure 1.

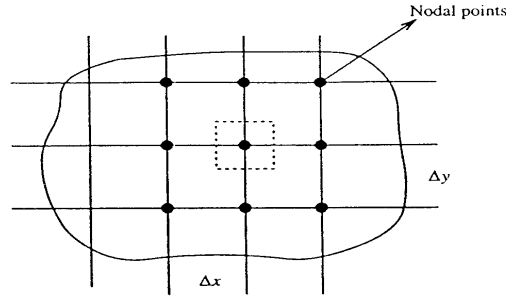


FIG. 1. Finite differencing along x and y

Each nodal point is designated by a numbering scheme i and j , where i indicates x increment and j indicates y increment, as shown in Figure 2. In a case study on temperature distribution, the temperature at each nodal point (x_i, y_j) is the average temperature of the surrounding hatched region.

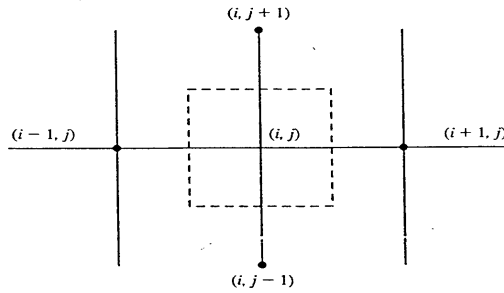


FIG. 2. 5-point stencil for Laplace equation

A finite difference equation suitable for the interior nodes of a steady two-dimensional system can be obtained by considering Laplace's equation at the nodal point i, j as

$$\frac{\partial^2 T}{\partial x^2} \Big|_{i,j} + \frac{\partial^2 T}{\partial y^2} \Big|_{i,j} = 0. \quad (1)$$

The second derivatives at the nodal point (i, j) can be approximated (derived from the Taylor series) as

$$\left. \frac{\partial^2 T}{\partial x^2} \right|_{i,j} \approx \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} \quad \text{and} \quad \left. \frac{\partial^2 T}{\partial y^2} \right|_{i,j} \approx \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}.$$

Equation (1) then gives

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0.$$

Assuming $\Delta x = \Delta y$, the finite difference approximation of Laplace's equation for interior regions can be expressed as $T_{i,j+1} + T_{i,j-1} + T_{i+1,j} + T_{i-1,j} - 4T_{i,j} = 0$ or

$$4T_{i,j} - T_{i-1,j} - T_{i,j-1} - T_{i+1,j} - T_{i,j+1} = 0 \quad (2)$$

More accurate higher order approximations for interior nodes and boundary nodes are also obtained in a similar manner.

Example. A two-dimensional rectangular plate ($0 \leq x \leq 1$, $0 \leq y \leq 1$) is subjected to the uniform temperature boundary conditions (with top surface maintained at 100°C and all other surfaces at 0°C) shown in Figure 3; that is, $T(0, y) = 0$, $T(1, y) = 0$, $T(x, 0) = 0$, and $T(x, 1) = 100^\circ\text{C}$.

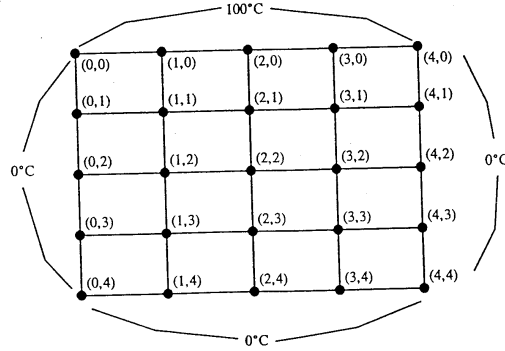


FIG. 3. Finite difference for a rectangular plate

Suppose we are interested only in the values of the temperature at the nine interior nodal points (x_i, y_j) , where $x_i = i\Delta x$ and $y_j = j\Delta y$, $i, j = 1, 2, 3$, with $\Delta x = \Delta y = \frac{1}{4}$. However, we assume symmetry for simplifying the problem. That is, we assume that $T_{3,3} = T_{1,3}$, $T_{3,2} = T_{1,2}$, and $T_{3,1} = T_{1,1}$. We thus have only six unknowns: $(T_{1,1}, T_{1,2}, T_{1,3})$ and $(T_{2,1}, T_{2,2}, T_{2,3})$. From equation (2), we then have:

$$\begin{aligned} 4T_{1,1} - 0 - T_{1,2} - T_{2,1} - 100 &= 0 \\ 4T_{2,1} - T_{1,1} - T_{2,2} - T_{1,1} - 100 &= 0 \\ 4T_{1,2} - 0 - T_{1,3} - T_{2,2} - T_{1,1} &= 0 \\ 4T_{2,2} - T_{1,2} - T_{2,3} - T_{1,2} - T_{2,1} &= 0 \\ 4T_{1,3} - 0 - 0 - T_{2,3} - T_{1,2} &= 0 \\ 4T_{2,3} - T_{1,3} - 0 - T_{1,3} - T_{2,2} &= 0 \end{aligned}$$

After suitable rearrangement, these equations can be written in the following form:

$$\begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 \\ -2 & 4 & 0 & -1 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & -1 & -2 & 4 & 0 & -1 \\ 0 & 0 & -1 & 0 & 4 & -1 \\ 0 & 0 & 0 & -1 & -2 & 4 \end{bmatrix} \begin{bmatrix} T_{1,1} \\ T_{2,1} \\ T_{1,2} \\ T_{2,2} \\ T_{1,3} \\ T_{2,3} \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solution of this system will give us temperatures at the nodal points.

In FDM, as we have seen above, the PDE is converted into a set of linear, simultaneous equations. When the simultaneous equations are written in matrix notation, the majority of the elements of the matrix are zero. Such matrices are called “sparse matrix”. However, for any meaningful problem, the number of simultaneous equations becomes very large, say of the order of a few thousand. There are special purpose routines that deal with very large, sparse matrices. Furthermore, one needs skillful ways of storing such large matrices, otherwise, several Gigabits will be used up just for the storing. An alternative way of solving very large system of simultaneous equations is iterative. The advantage of iterative solution is that the storing of large matrices is unnecessary.