Derivation of Transfer Function for Capacitor

Starting with Kirchhoff's Law,

$$V_{in}(t) = Ri(t) + \frac{1}{C}\int i(t)dt \tag{1}$$

transform Eq. 1 into Laplace domain,

$$V_{in}(s) = Ri(s) + \frac{1}{sC}i(s) \tag{2}$$

Solve for current $i(s)$

$$i(s) = \frac{V_{in}(s)}{R + \frac{1}{sC}} \tag{3}$$

then calculate the voltage across capacitor,

$$V_c(s) = \frac{1}{sC}i(s) \tag{4}$$

inserting Eq. 3 into Eq. 4

$$V_c(s) = \frac{1}{sC}\frac{V_{in}(s)}{R + \frac{1}{sC}} \tag{5}$$

arranging the terms will yield,

$$V_c(s) = \frac{V_{in}(s)}{1 + sRC} \tag{6}$$

substitute $s = j\omega$ to transfer the solution to frequency domain

$$\frac{V_{c-out}}{V_{in}} = \frac{1}{1 + j\omega RC} \tag{7}$$

multiplying numerator and denominator by denominator's complex conjugate

$$\frac{V_{c-out}}{V_{in}} = \frac{1 - j\omega RC}{1 + (\omega RC)^2} \tag{8}$$

Eq. 8 is a complex number, having both real and imaginary parts. Now, we will calculate both amplitude and phase of this equation. In general, a complex number $a + jb$ can be expressed in terms of its amplitude A and phase ϕ as

$$|A| = \sqrt{a^2 + b^2}$$

$$\phi = \arctan\left(\frac{b}{a}\right)$$

By using above identities, amplitude and phase of the transfer function when the output measured across the capacitor can be calculated as,

$$\left|\frac{V_{c-out}}{V_{in}}\right| = \frac{1}{\sqrt{1 + (\omega RC)^2}} \tag{9}$$

$$\phi = -\arctan(\omega RC) \tag{10}$$