different ways to find the inverse transform.

HW5-1

* 7.3 Use the properties tabulated in Appendix B to find the Laplace transform of each of the following functions of time.

a) $f_1(t) = t\epsilon^{-2t}\cos 3t$

b) $f_2(t) = t^2 \sin 2t$

c) $f_3(t) = \frac{d}{dt}(t^2\epsilon^{-t})$

d) $f_4(t) = \int_0^t \lambda^2 \epsilon^{-\lambda} d\lambda$

a) $f_1(t) = t\,e^{-2t}\cos 3t$

From tables if $f = e^{-2t}\cos 3t$ then $F(s) = \dfrac{s+2}{(s+2)^2+9}$

$f_1 = t\,f(t) \Rightarrow$

$$F_1(s) = -\frac{d}{ds}F(s) = -\left[\frac{1}{(s+2)^2+9} - \frac{(s+2)\,2\,(s+2)}{[(s+2)^2+9]^2}\right]$$

$$F_1(s) = -\left[\frac{(s+2)^2+9 - 2(s+2)^2}{[(s+2)^2+9]^2}\right]$$

$$= -\left[\frac{9-(s+2)^2}{[(s+2)^2+9]^2}\right]$$

$$F_1(s) = \frac{s^2+4s-5}{[s^2+4s+13]^2}$$

b) $f_2(t) = t^2\sin 2t = \dfrac{t^2}{2j}\left[e^{j2t} - e^{-j2t}\right]$

From tables

$$F_2(s) = \frac{1}{\cancel{2}j}\left[\frac{\cancel{2}}{s-2j} - \frac{\cancel{2}}{s+2j}\right] = \frac{1}{j}\cdot\left[\frac{\cancel{s}+2j-\cancel{s}+2j}{s^2+4}\right]$$

$$F_2(s) = \frac{4}{s^2+4}$$

c) $f_3(t) = \frac{d}{dt}\left(t^2 e^{-t}\right)$

Let $f = t^2 e^{-t}$ Tables $\Rightarrow$ $F(s) = \frac{2}{(s+1)^2}$

$\frac{df}{dt} = f_3 \Rightarrow F_3(s) = sF(s) - \cancel{f(0)}$ 0

$\therefore F_3(s) = \frac{2s}{(s+1)^2}$

d) $f_4(t) = \int_0^t \lambda^2 e^{-\lambda} dt$

Let $f(t) = t^2 e^{-t}$. Tables $\Rightarrow$ $F(s) = \frac{2}{(s+1)^2}$

$f_4 = \int_0^t f(\lambda)\, d\lambda \Rightarrow F_4(s) = \frac{1}{s} F(s)$ (Tables)

$\therefore F_4(s) = \frac{2}{s(s+1)^2}$

7.5 a) $F(s) = \frac{2s^3 + 3s^2 + s + 4}{s^3}$

b) $F(s) = \frac{3s^2 + 9s + 24}{(s-1)(s+2)(s+5)}$

c) $F(s) = \frac{4}{s^2(s+1)}$

d) $F(s) = \frac{3s}{s^2 + 2s + 26}$

a) Given $F(s) = \dfrac{2s^3 + 3s^2 + s + 4}{s^3}$

Divide

$$F(s) = 2 + \frac{3}{s} + \frac{1}{s^2} + \frac{4}{s^3}$$

Tables ⇒

$$f(t) = 2\delta(t) + 3H(t) + t + 2t^2 \quad \text{for } t > 0.$$

b) Given $F(s) = \dfrac{3s^2 + 9s + 24}{(s-1)(s+2)(s+5)}$

Distinct Poles

$$F(s) = \frac{A_1}{s-1} + \frac{A_2}{s+2} + \frac{A_3}{s+5}$$

$$A_1 = \left.\frac{3s^2 + 9s + 24}{(s+2)(s+5)}\right|_{s=1} = \frac{3+9+24}{3 \cdot 6} = \frac{36}{18} = 2$$

$$A_2 = \left.\frac{3s^2 + 9s + 24}{(s-1)(s+5)}\right|_{s=-2} = \frac{12 - 18 + 24}{-3 \cdot 3} = \frac{18}{-9} = -2$$

$$A_3 = \left.\frac{3s^2 + 9s + 24}{(s-1)(s+2)}\right|_{s=-5} = \frac{75 - 45 + 24}{-6 \cdot -3} = \frac{6}{18} = \frac{1}{3}$$

$$\therefore F(s) = \frac{2}{s-1} - \frac{2}{s+2} + \frac{1/3}{s+5}$$

$$\Rightarrow f(t) = 2e^{t} - 2e^{-2t} + \frac{1}{3}e^{-5t}$$

7.5 (c) $\quad F(s) = \dfrac{4}{s^2 (s+1)}$

Consider $\quad G(s) = 4/(s+1) \overset{\text{Tables}}{\Longrightarrow} g(t) = 4e^{-t}$

Then $\quad f(t) = \iint g(t)\, dt.$

i.e. $\quad \displaystyle\int_0^t g(\tau)\, d\tau = \int_0^t 4 e^{-\tau} d\tau = 4 \left.\frac{e^{-\tau}}{-1}\right|_0^t$

$$= -4\left(e^{-t} - 1\right)$$

$$f(t) = 4\int_0^t \left(1 - e^{-\tau}\right) d\tau = 4\left(\tau + e^{-\tau}\right)\Big|_0^t$$

$$f(t) = 4\left(t + e^{-t} - 1\right)$$

d) $\quad F(s) = \dfrac{3s}{s^2 + 2s + 26}$

Complete the square in the denominator.

$$s^2 + 2s + 26 = (s+1)^2 + 25$$

$$\therefore F(s) = \frac{3s}{(s+1)^2 + 25} = \frac{3(s+1)}{(s+1)^2 + 25} - \frac{3}{(s+1)^2 + 25}$$

From tables

$$f(t) = 3e^{-t}\cos 5t - \frac{3}{5}e^{-t}\sin 5t$$

$$= 3e^{-t}\left[\cos 5t - \frac{1}{5}\sin 5t.\right]$$

Additional Problems.

1. Use Laplace transforms to find y(t) where y(0) = 0, y'(0) = 0 and
$m\,y''(t) + k\,y(t) = f_0\,\delta(t)$

$$m\left[-\underbrace{\cancel{y'(0)}}_{0} - s\underbrace{\cancel{y(0)}}_{0} + s^2 Y(s)\right] + k\,Y(s) = f_0$$

$$\Rightarrow Y(s) = \frac{f_0}{m s^2 + k} = \frac{1}{m}\,\frac{f_0}{s^2 + \frac{k}{m}}$$

This is of the form: $Y(s) = \frac{f_0}{m}\frac{1}{\omega}\qquad\frac{\omega}{s^2+\omega^2}$

where $\omega = \sqrt{\frac{k}{m}}$. From the tables

$$y(t) = \frac{f_0}{m}\frac{1}{\omega}\sin\omega t \qquad \omega = \sqrt{\frac{k}{m}}$$

2. Suppose y(t) satisfies the equation:
$y''(t) + y'(t) + 25\,y(t) = f(t)$
subject to the initial conditions: $y(0) = 0$ and $y'(0) = 0$.

Use Laplace transforms to do the following exercises

a. Find and plot the solution, $y_I(t)$, when f(t) is the unit impulse $\delta(t)$.

b. At what value of time t_0 is y_I first equal to zero? ($t_0 > 0$).

c. Find and plot the solution, $y_H(t)$, when f(t) is the unit step function H(t).

d. From the plot determine the time t_M at which y_H is maximum. What is the maximum value of y_H? How does t_M compare with t_0 from part b?

e. Explain the results that you got in part d.

2 a Take Laplace Transform. For $y(0)=y'(0)=0$ we get.

$$(s^2+s+25)\,Y_I(s) = 1$$

$$\Rightarrow \quad Y_I(s) = \frac{1}{s^2+s+25}$$

Complete the square

$$Y_I(s) = \frac{1}{(s+\frac{1}{2})^2+24.75}$$

This is of the form $\dfrac{\omega}{(s+1/2)^2+\omega^2} \cdot \dfrac{1}{\omega}$

where $\omega = \sqrt{24.75}$

$$y_I(t) = \frac{e^{-t/2}}{\omega} \sin \omega t.$$ See plot on attached mathcad sheet

b. $t_0 = \dfrac{\pi}{\omega} = 0.63$

c. In this case

$$Y_H(s) = \frac{1}{s} Y_I(s) \Rightarrow$$

$$y_H(t) = \int_0^t y_I(\tau)\,d\tau$$

Note that $y_I(t)$ has the form $c\,e^{-at} \sin \omega t$

Note to grader: Give full credit for looking up integral

Integrate by parts   8 extra credit for analytical solution

$$I = \int_0^t e^{-a\tau}\sin\omega\tau\,d\tau = -e^{-a\tau}\frac{\cos\omega\tau}{\omega}\Big|_0^t - \int_0^\tau \frac{1}{\omega}\cos\omega\tau\, a\, e^{-a\tau}\,d\tau$$

$$u = e^{-a\tau} \qquad du = -a e^{-a\tau}$$

$$dv = \sin\omega\tau\, d\tau \Rightarrow v = -\frac{\cos\omega\tau}{\omega}$$

$\int$ by parts again

$$u = e^{-a\tau} \qquad du = -a\, e^{-a\tau}$$

$$dv = \cos\omega\tau\, d\tau \qquad v = \frac{\sin\omega\tau}{\omega}$$

$$I = -e^{-a\tau}\frac{\cos\omega\tau}{\omega}\Big|_0^t - \frac{a}{\omega}\left[\frac{e^{-a\tau}}{\omega}\sin\omega\tau\Big|_0^\tau + \frac{a}{\omega}\underbrace{\int_0^t \sin\omega\tau\, e^{-a\tau}\,d\tau}_{I}\right]$$

$$\therefore \left(1 + \frac{a^2}{\omega^2}\right) I = \frac{1}{\omega}\left(1 - e^{-at}\cos\omega t\right) + \frac{a}{\omega^2}\left(-e^{-at}\sin\omega t\right)$$

$$\therefore I = \frac{\omega^{\cancel{2}}}{\omega^2+a^2}\left[\frac{1}{\cancel{\omega}}\left(1 - e^{-at}\left(\cos\omega t + \frac{a}{\omega}\sin\omega t\right)\right)\right]$$

$$\therefore y_H(t) = \frac{1}{\omega^2+a^2}\left[1 - e^{-at}\left(\cos\omega t + \frac{a}{\omega}\sin\omega t\right)\right]$$

where $\omega = \sqrt{24.75}$ and $a = \frac{1}{2}$

Since $y_I(t) = \frac{d}{dt}y_H(t)$, the maximax value of y_H occurs at t_0. $y_H(t_0) = 0.069$ (see Mathcad result).

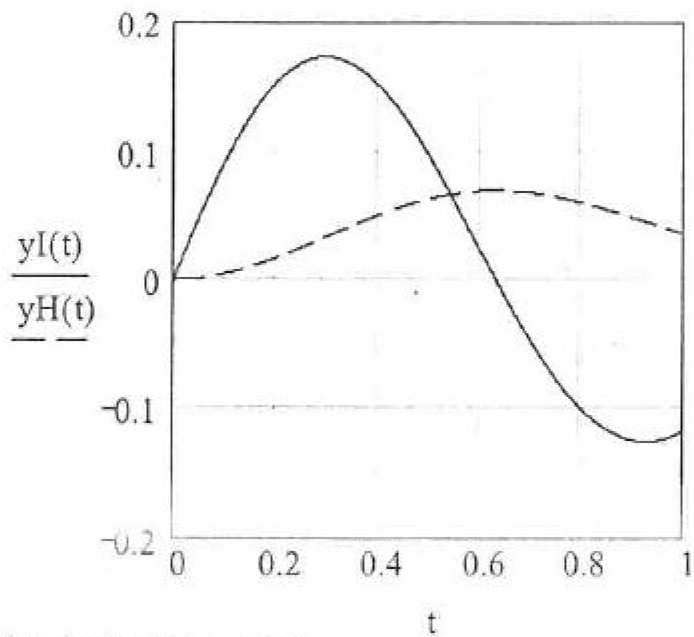
Define Constants

$$a := 0.5 \qquad w := \sqrt{24.75}$$

Define Solutions

$$yI(t) := e^{-a\cdot t}\cdot\frac{\sin(w\cdot t)}{w}$$

$$yH(t) := \left[1 - e^{-a\cdot t}\cdot\left(\cos(w\cdot t) + \frac{a\cdot\sin(w\cdot t)}{w}\right)\right]\cdot\frac{1}{w^2 + a^2}$$



Check Analytical Results

$$to := \frac{\pi}{w} \qquad yH(to) = 0.0691699 \qquad yI(to) = 0$$

$$yH(to + .001) = 0.0691695 \qquad yH(to - .001) = 0.0691695$$