1.

1. Consider the mass/spring system of Figure 1. Assume $y(t) = 0$. Let $M = 1000$ kg and $K = 40{,}000$ N/m. Assume $f(t)$ is from gravity, $x(0) = 0.1$ and $\dot{x}(0) = 1\,\text{m/s}$.
   a. What is the natural frequency of the system in radians/second and in Hertz.
   b. Find $x(t)$. Plot $x(t)$ for $0 < t < 2$. What is the maximum value of x over this interval?

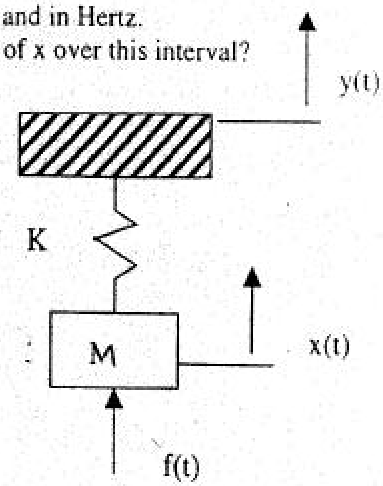


Figure 1

a) Natural frequency in radians/sec.
From class

$$\omega_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{40\,000}{1000}} = \sqrt{40}$$

$$\omega_0 = 6.324 \text{ rad/sec}$$

Hertz:

$$f_0 = \frac{\omega_0}{2\pi} = 1.00658$$

b. $$M\ddot{x}(t) + Kx(t) = -gM$$

$$x = x_h + x_p$$

Assume $x_p = C \Rightarrow KC = -gM \Rightarrow C = -\frac{gM}{K}$

$$\therefore\ x_p = \frac{-9.8 \cdot 1000}{40\,000} = -0.245\,m$$

x_h: From class

$$x_h(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

Apply initial conditions to solve for A and B

$$x(0) = x_h(0) + x_p = A - \frac{gm}{K} = .1 \Rightarrow A = .1 + \frac{gm}{K}$$

$$A = .345$$

$$\dot{x}(0) = 1 = B \cdot \omega_0 \Rightarrow B = \frac{1}{6.324} = 0.158$$

$\therefore \quad x(t) = 0.345 \cos \omega_0 t + 0.158 \sin \omega_0 t - 0.245$

See MathCad plot on page HW2-3.

$x_{max} = 0.134$ (from plot)

or analytically $x_{max} = \sqrt{0.345^2 + 0.158^2} - 0.245$

2. Consider the system of Figure 1. If f(t) = 4000 cos(ωt)
   a. Find the particular solution to the governing differential equation. Determine the magnitude of the amplitude, R, and the phase of the response, ϕ, as functions of the excitation frequency ω.
   b. Plot R(ω) and ϕ(ω) for 0 < ω < 20. Indicate on your plot of R the regions that are primarily controlled by 1) the spring and 2) the mass.

From class, if $f = f_0 \cos\omega t$ then $x = \dfrac{f_0}{K - M\omega^2} \cos\omega t$

The $R = \left| \dfrac{f_0}{K - M\omega^2} \right| = \left| \dfrac{4000}{40000 - 1000\omega^2} \right|$

$$\phi = \begin{cases} 0 & \omega < \omega_0 \\ \pi & \omega > \omega_0 \end{cases}$$

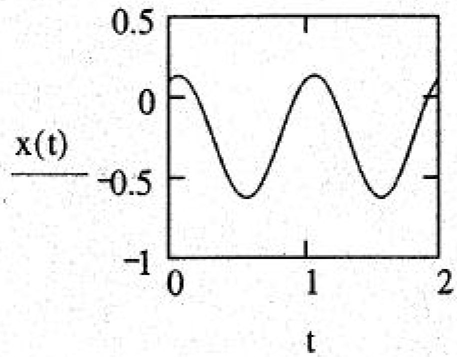
See plots on HW 2-3

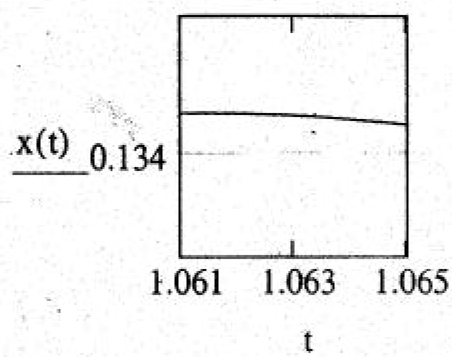
From class, the response is controlled by K if $\omega < .3\omega_0 \approx 2$ and is controlled by M if $\omega > \sqrt{2}\,\omega_0 \approx 9$.

1. b) $K := 40000 \quad M := 1000 \quad \omega o := \sqrt{\frac{K}{M}}$

$\omega o = 6.325$

$x(t) := 0.345 \cdot \cos(\omega o \cdot t) + 0.158 \cdot \sin(\omega o \cdot t) - .245$





$x(1.06) = 0.13444$

$t := 1.06 \qquad tmax := \mathrm{maximize}(x, t) \qquad tmax = 1.061$

$x(tmax) = 0.134459$

Alternatively $\quad xmax := \sqrt{(.345)^2 + (.158)^2} - .245$

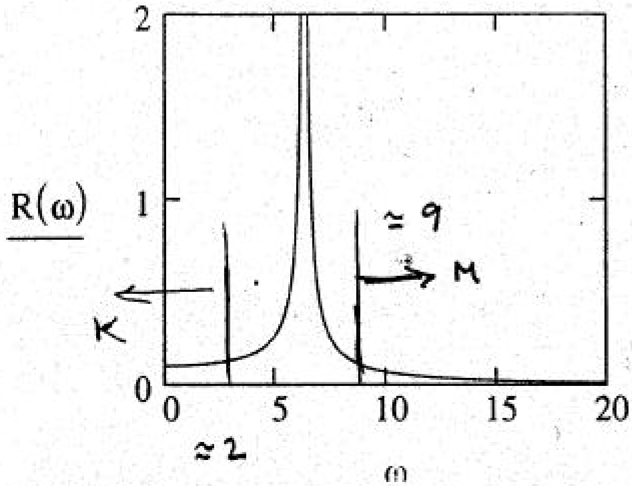
$xmax = 0.134459$

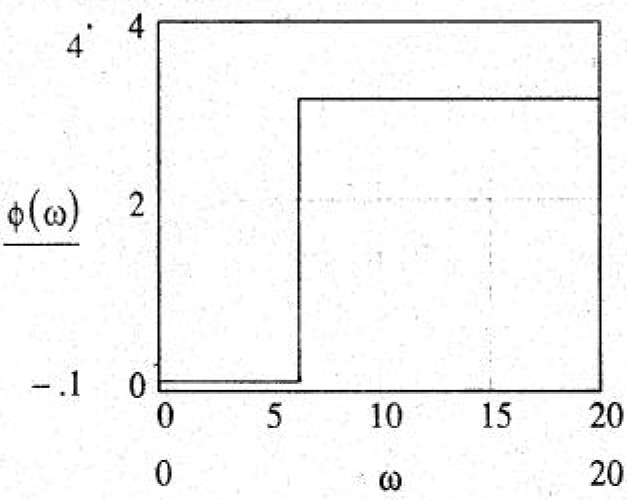
2. b)

$$R(\omega) := \left| \frac{4000}{40000 - 1000 \cdot \omega^{2}} \right|$$

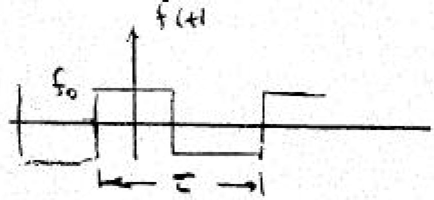
$$\phi(\omega) := \begin{vmatrix} 0 & \text{if } \omega < \omega o \\ \pi & \text{if } \omega > \omega o \end{vmatrix}$$

$$\phi(8) = 3.142$$





3. Assume that the force f(t) is the square wave shown in Figure 2



$f_0 = 1000$ N

$\tau = 3$ sec

Figure 2

a. Find the Fourier series for f(t), i.e. find F_n and G_n where

$$f(t) = \frac{F_0}{2} + \sum_{n=1}^{\infty} F_n \cos(n\omega t) + G_n \sin(n\omega t)$$

b. Find the particular solution if the square wave is applied as a force to the system of Figure 1, i.e. assume that x(t) has the same period as f(t) and can be represented as a Fourier series of the form

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

Find the a_n and b_n. Which frequency component of the response is the largest? Why?

$$\frac{F_0}{2} = \bar{f}(t) \quad (\text{average}) = 0$$

$$f(t) = f(-t), \text{ even function} \Rightarrow G_n = 0$$

$$f(t) \text{ even} \Rightarrow \quad F_n = \frac{4}{\tau}\int_0^{\tau/2} f(t)\cos(n\omega t)\,dt$$

$$f(t) = \begin{cases} f_0 & t < \tau/4 \\ -f_0 & \frac{\tau}{4} < t < \frac{\tau}{2} \end{cases}$$

$$F_n = \frac{4}{\tau} f_0 \left[\int_0^{\tau/4} \cos n\omega t\, dt - \int_{\tau/4}^{\tau/2} \cos n\omega t\, dt \right]$$

$$= \frac{4}{\tau} f_0 \left[\left.\frac{\sin n\omega t}{n\omega}\right|_0^{\tau/4} - \left.\frac{\sin n\omega t}{n\omega}\right|_{\tau/4}^{\tau/2} \right]$$

$$= \frac{4}{\tau}\frac{f_0}{n\omega} \left[\sin n\omega\frac{\tau}{4} - \sin n\omega\frac{\tau}{2} + \sin n\omega\frac{\tau}{4} \right]$$

Now $\omega = \frac{2\pi}{\tau}$. So $\sin \frac{n\omega\tau}{4} = \sin\frac{n\pi}{2}$

and $\sin n\frac{\omega\tau}{2} \doteq \sin n\pi = 0$

$$F_n = \frac{4f_0}{\cancel{\tau} n \frac{2\pi}{\cancel{\tau}}} \cdot 2\sin\frac{n\pi}{2} = \frac{4f_0}{n\pi}\sin\frac{n\pi}{2}$$

b. From part a $\quad f(t) = \sum_{n=1}^{\infty} F_n \cos(n\omega t)$

From class $\quad x(t) = \sum_{n=1}^{\infty} \frac{F_n}{K - M(n\omega)^2} \cos(n\omega t)$

Thus $a_n = \frac{F_n}{K-M(n\omega)^2}$ $\quad a_0 = 0$ and $b_n = 0$.

See mathcad evaluation of a_n. a_3 has largest magnitude because $3\cdot\omega = \omega_0$.

4. Suppose that instead of a force being applied to the system in Figure 1 the system is excited by a sinusoidal motion at y. That is $y(t) = Y_0 e^{j\omega t}$. Find the particular solution. Use your result to determine the particular solution if instead $y(t) = Y_0 \sin(\omega t)$.

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The equation of motion is: $M\ddot{x} = -K(x-y)$

$\Rightarrow M\ddot{x} + Kx = Ky = KY_0 e^{j\omega t}$

Assume $x = Ae^{j\omega t}$

$\Rightarrow \ddot{x} = -\omega^2 A e^{j\omega t} \Rightarrow (K - M\omega^2)Ae^{j\omega t} = KY_0 e^{j\omega t}$

$\Rightarrow A = KY_0/(K - M\omega^2)$

But $y(t) = Y_0 e^{j\omega t} = Y_0 \cos\omega t + j\, Y_0 \sin\omega t$

$\Rightarrow$ $Im(x(t))$ is the response to $Y_0 \sin\omega t$

$$Im\,(x(t)) = Im\left(\frac{kY_0}{K - M\omega^2} \cdot (\cos\omega t + j\sin\omega t)\right)$$

$$= \frac{kY_0}{K - M\omega^2} \sin\omega t .$$