A.

1. The transfer function is

$$T(s) = \frac{KK_m}{s + 1/\tau_m} \cdot \frac{1}{1 + \frac{KK_m}{s + 1/\tau_m}} = \frac{KK_m}{s + \frac{1}{\tau_m} + KK_m}$$

Pole is in the left hand so we can use the final value theorem.

$$\Omega_o(s) = \Omega_i(s) \cdot T(s) = \frac{1}{s} \frac{KK_m}{s + \frac{1}{\tau_m} + KK_m}$$

Final value theorem:

$$\omega_o(\infty) = \lim_{s \to 0} s\Omega_o(s) = \frac{KK_m}{\frac{1}{\tau_m} + KK_m} < 1$$

want $\displaystyle \frac{|\omega_o(\infty) - \omega_i|}{\omega_i} = .05$

$$\omega_i = 1 \Rightarrow 1 - \frac{KK_m}{\frac{1}{\tau_m} + KK_m} = 0.05$$

$$\text{or} \quad \frac{KK_m}{\frac{1}{\tau_m} + KK_m} = 0.95$$

$$\Rightarrow \frac{1}{\frac{1}{\tau_m KK_m} + 1} = 0.95$$

$$\Rightarrow 1 = 0.95\left(\frac{1}{\tau_m KK_m} + 1\right)$$

$$\Rightarrow \frac{1}{\tau_m KK_m} = \frac{0.05}{0.95} = 0.053$$

$$\Rightarrow \quad \tau_m K K_m = 19$$

$$\therefore \quad K = \frac{19}{\tau_m K_m} = \frac{19}{150 \cdot 0.4} = 0.32$$

Motor speed as a function of time:

$$\Omega_0(s) = \frac{K K_m}{s(s+a)} \qquad a = \frac{1}{\tau_m} + K \cdot K_m$$

Partial fractions

$$\Omega_0 = \frac{A}{s} + \frac{B}{s+a}$$

$$\Rightarrow \quad A = \frac{K K_m}{a}$$

$$B = \frac{K K_m}{-a}$$

$$\Rightarrow \quad \omega_0(t) = \frac{K K_m}{a}\left(1 - e^{-at}\right)$$

where $a = \frac{1}{.4} + 150 \cdot (0.32) = 50$, $K K_m = 47.5$

$$\omega_0(t) = 0.95\left(1 - e^{-50t}\right)$$

2. The transfer function is the same as in part 1 except we need to replace K with K/s $\Rightarrow$

$$T(s) = \frac{K}{s}\,\frac{K_m}{\left(s + \frac{1}{\tau_m} + \frac{K}{s} K_m\right)} = \frac{K K_m}{s^2 + \frac{s}{\tau_m} + K K_m}$$

From class we know that the damping ratio and natural frequency are

$$\omega_0 = \sqrt{K \cdot K_m} \qquad (1)$$

$$\zeta = \frac{1/\tau_m}{2\,\omega_0} \qquad (2)$$

Also the fractional overshoot is given by the equation

$$OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \qquad (3)$$

Thus, we want $e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.05$

$$\Rightarrow \frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{3}{\pi} = 0.95$$

$$\Rightarrow \frac{\zeta^2}{1-\zeta^2} = 0.91 \quad \Rightarrow \quad \zeta^2 = .91 - .91\zeta^2$$

$$\Rightarrow 1.91\zeta^2 = .91 \quad \Rightarrow \zeta = 0.69 \qquad (4)$$

$$\therefore (1),(2),(4) \Rightarrow \frac{1/\tau_m}{2\sqrt{K \cdot K_m}} = 0.69$$

$$\Rightarrow \sqrt{KK_m} = \frac{1}{\tau_m \cdot 2 \cdot 0.69} = 1.81$$

.4

$$\Rightarrow K = \frac{3.28}{K_m} = \frac{3.28}{150} = 0.0219$$

Dynamic Response:

Then $$T(s) = \frac{(0.0219)(150)}{s^2 + 2.5s + (0.0219)(150)}$$

$$= \frac{3.28}{s^2 + 2.5 \cdot s + 3.28}$$

Laplace transform of output

$$\Theta_o(s) = \frac{1}{s} T(s) = \frac{1}{s} \cdot \frac{3.28}{s^2 + 2.5s + 3.28}$$

Let $\Theta_o(s) = \frac{A}{s} + F(s)$

$$A = 1 \Rightarrow F(s) = \frac{3.28 - (s^2 + 2.5s + 3.28)}{s(s^2 + 2.5s + 3.28)}$$

$$= -\frac{s + 2.5}{s^2 + 2.5s + 3.28}$$

Complete the square for the denominator.

$$s^2 + 2.5s = (s + 1.25)^2 - (1.25)^2$$

$$3.28 - 1.25^2 = 1.72$$

$$\sqrt{1.72} = 1.31 \equiv \omega$$

$$\therefore F(s) = -\frac{s + 2.5}{(s + 1.25)^2 + \omega^2}$$

The Laplace transform tables in book ⇒

(Note $B = -1$, $C = -2.5$, $\omega = 1.31$, $a = 1.25$

$$f(t) = -\left[\cos\omega t + \left(\frac{2.5 - 1.25}{1.31}\right)\sin\omega t\right]e^{-1.25t}$$

Dynamic Response

$$\theta_o(t) = 1 - e^{-1.25t}\left(\cos\omega t + 0.95\sin\omega t\right)$$

$\omega = 1.31$

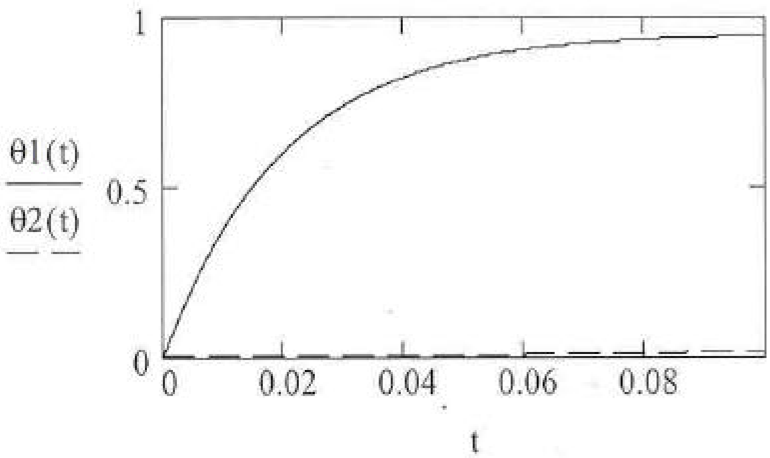
See plots from Mathcad on next page.

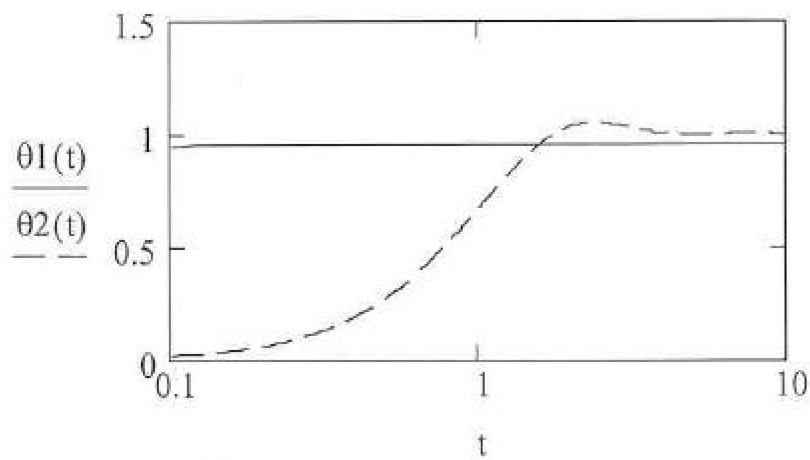
Proportional Control

$$\theta1(t) := 0.95\cdot\left(1 - e^{-50\cdot t}\right)$$

Integral Control $\quad \omega := 1.31$

$$\theta2(t) := 1 - e^{-1.25\cdot t}\cdot\left(\cos(\omega\cdot t) + 0.95\cdot\sin(\omega\cdot t)\right)$$





B.

- Equations of motion

$$m\ddot{x}_1 = -(1)(x_1 - x_2) - (1)\cdot x_1 \quad = \quad -2x_1 + x_2$$

$$m\ddot{x}_2 = -(1)(x_2 - x_1) \quad = \quad -x_2 + x_1$$

In matrix form

$$m\ddot{\vec{x}} + \mathbb{K}\vec{x} = \vec{0} \qquad \mathbb{K} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Assume $\vec{x} = \vec{b}\, e^{j\omega t}$ (or $\vec{x} = \vec{b}\sin\omega t$ or $\vec{b}\cos\omega t$)

$$\ddot{\vec{x}} = -\omega^2 \vec{b}\, e^{j\omega t}$$

$$\therefore \quad -m\omega^2\vec{b}\,\cancel{e^{j\omega t}} + \mathbb{K}\vec{b}\,\cancel{e^{j\omega t}} = 0$$

$$\Rightarrow \mathbb{K}\vec{b} = m\omega^2\vec{b}$$

or $\quad \left[\mathbb{K} - m\omega^2\mathbb{I}\right]\vec{b} = \vec{0}$

Either $\vec{b} = \vec{0}$ or $\det\left[\mathbb{K} - m\omega^2\mathbb{I}\right] = 0$

$m = 1$

$$\Rightarrow \begin{vmatrix} (2-\omega^2) & -1 \\ -1 & (1-\omega^2) \end{vmatrix} = 0 \Rightarrow (2-\omega^2)(1-\omega^2) - 1 = 0$$

This implies $\quad \omega^4 - 3\omega^2 + 2 - 1 = 0$

or $\quad \omega^4 - 3\omega^2 + 1 = 0$

Quadratic formula $\Rightarrow \omega^2 = \dfrac{3 \pm \sqrt{9 - 4\cdot 1}}{2} = \dfrac{3 \pm \sqrt{5}}{2}$

Mode shape for lower frequency

$$\omega^2 = \omega_1^2 = \frac{3-\sqrt{5}}{2} = 0.382 \Rightarrow \omega_1 = 0.618$$

$$(2-\omega_1^2)\, b_1 - b_2 = 0$$

$$\Rightarrow \left(2 - \frac{3-\sqrt{5}}{2}\right) b_1 = b_2$$

$$\Rightarrow (2 - .382)\, b_1 = b_2 \Rightarrow b_2 = 1.618$$

$$\therefore \vec{b}_1 = c_1 \begin{bmatrix} 1 \\ 1.618 \end{bmatrix}$$ (Note: any multiple of this vector is correct)

Mode shape for higher frequency: $\omega^2 = \omega_2^2 = \frac{3+\sqrt{5}}{2} = 2.618$

$\omega_2 = 1.618$

$$(2-\omega_2^2)\, b_1 - b_2 = 0$$

$$(2 - 2.618)\, b_1 = b_2 \Rightarrow b_2 = -0.618\, b_1$$

$$\vec{b}_2 = c_2 \begin{bmatrix} 1 \\ -0.618 \end{bmatrix}$$ (Note: any multiple of this vector is correct.)