14.1 This problem involves the servomechanism considered in Section 14.1, and the numerical values of the parameters given in Table 14.1 should be used where they are needed.

a) Calculate the value of K_A for a proportional-only controller that will result in a damping ratio of $\zeta = 0.7071$ for the closed-loop system. Give its transfer function $T(s)$ as a ratio of polynomials.

b) Assuming that a tachometer signal $e_3 = K_T\phi$ is available for proportional-plus-derivative control, calculate the values of K_A and K_T that will yield a closed-loop system with $\zeta = 0.7071$ and $\omega_n = 8$ rad/s. Determine the closed-loop transfer function ~~and verify that its poles have the specified values for ζ and ω_n.~~

—

From (9), the transfer function is

$$T(s) = \frac{a K_A}{s^2 + (1/\tau_m)s + aK_A} \quad \text{where } a = \frac{K_m K_\theta}{N}$$

From Table 14.1

$$\begin{aligned} K_\theta &= 15/\pi \\ K_m &= 150 \\ N &= 10 \\ \tau_m &= 0.4 \end{aligned} \Rightarrow a = 71.6$$

From class we know that if the transfer function has a denominator of the form

$$s^2 + a_1 s + a_0 \quad \text{then} \quad \omega_0 = \sqrt{a_0}$$

$$\zeta = \frac{1}{2\tau_m \sqrt{71.6 K_A}} \qquad \zeta = \frac{a_1}{2\sqrt{a_0}}$$

$$\zeta = \frac{1}{\sqrt{2}} \Rightarrow \sqrt{K_A} = \frac{\sqrt{2}}{2\tau_m \cdot \sqrt{71.6}}$$

$$\text{or} \quad K_A = \frac{1}{2\tau_m^2 \cdot 71.6} = \frac{1}{2(0.16)\cdot 71.6} \approx 0.0436$$

Its transfer function becomes (substitute numerical values)

$$T(s) = \frac{71.6\cdot(.0436)}{s^2 + \frac{1}{.4}s + 71.6\cdot(0.0436)}$$

$$= \frac{3.13}{s^2 + 2.5s + 3.13}$$

b) Use derivation + proportional control to give $\zeta = 0.7071$ & $\omega_n = 8$

From textbook, equation (14) ⇒

$$T(s) = \frac{K_A K_m K_\theta / N}{s^2 + \left[\frac{1}{\tau_m} + K_A K_m K_T\right] s + \frac{K_A K_m K_\theta}{N}}$$

$$\omega_0 = 8 = \sqrt{\frac{K_A K_m K_\theta}{N}} \Rightarrow K_A = \frac{64\cdot N}{K_m K_\theta} \approx \frac{64\cdot 10\cdot \pi}{150\cdot 15}$$

$$K_A = 0.894$$

$$\zeta = \frac{\frac{1}{\tau_m} + K_A K_m K_T}{2\omega_0} \Rightarrow 2\omega_0\zeta - \frac{1}{\tau_m} = K_A K_m K_T$$

or $$K_T = \frac{2\omega_0\zeta - \frac{1}{\tau_m}}{K_A K_m} = \frac{2\cdot 8\cdot(.7071) - \frac{1}{.4}}{0.894\cdot 150}$$

$$K_T = 0.0657$$

From (14) the closed loop transfer function is

$$T(s) = \frac{64}{s^2 + \left(\frac{1}{.4} + 0.894 \cdot 150 \cdot 0.0657\right) s + 64}$$

$$T(s) = \frac{64}{s^2 + 11.3\,s + 64}$$

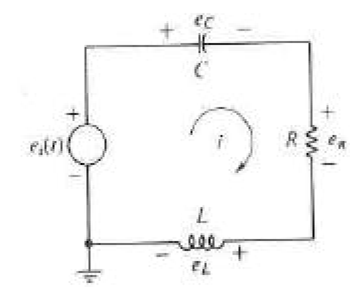
Note to grader: Students <u>do not</u> have to verify that the poles have the specified value for ζ & ω_n

14.3 Repeat the analysis of Example 14.2 for the series RLC circuit with a voltage source that was studied in Example 5.1.

a) Write the transfer function $T(s) = E_R(s)/E_i(s)$, and find its poles in terms of R, L, and C.

b) Assuming that L and C are fixed, find the locus of the poles of $T(s)$ as the resistance R is increased from zero toward infinity. Sketch the locus in the s-plane.

c) Give the value R for which $T(s)$ has a repeated pole ($\zeta = 1$), and give the location of this pole in the s-plane.



For example 5.1, we are given the governing ODE (page 149)

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}\, i = \dot{e}_i$$

$e_r(t) = R \cdot i(t)$

Laplace transforms with zero initial conditions $\Rightarrow$

$$I(s)\left(Ls^2 + Rs + \frac{1}{C}\right) = sE_i(s)$$

$$\Rightarrow \quad \frac{E_o(s)}{E_i(s)} = \frac{R\,I(s)}{E_i(s)} = \frac{s}{Ls^2 + Rs + \frac{1}{C}}$$

Poles are at
$$s = \frac{-R}{2L} \pm \frac{\sqrt{R^2 - \frac{4L}{C}}}{2L} \qquad (1)$$

or

$$s = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \qquad (2)$$

if $\frac{R^2}{4L^2} < \frac{1}{LC}$ or $R < 2\sqrt{\frac{L}{C}}$

Fix L & C let $R = 0$. Poles are at $s = \pm j\sqrt{\frac{1}{LC}}$

For $0 < R < 2\sqrt{\frac{L}{C}}$ we have a pair of complex conjugate poles in the left hand plane. The radius is

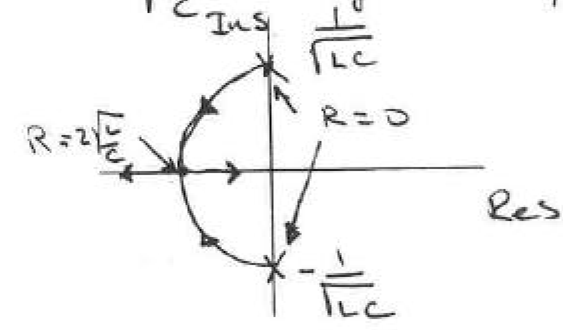
$$r_0 = \sqrt{\cancel{\frac{R^2}{4L^2}} + \frac{1}{LC} - \cancel{\frac{R^2}{4L^2}}} = \sqrt{\frac{1}{LC}}$$

c) When $R = 2\sqrt{\frac{L}{C}}$ we get a double pole at $s = -\sqrt{\frac{1}{LC}}$.

For $R > 2\sqrt{\frac{L}{C}}$ we get two poles on the negative axis

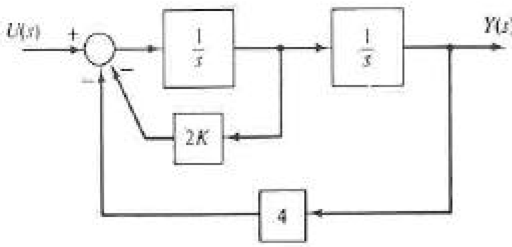
See figure.

b).



*14.5 a) Find the closed-loop transfer function $Y(s)/U(s)$ in terms of the parameter K for the feedback system shown in Figure P14.5.

b) Write an expression for the closed-loop poles in terms of K, and sketch the locus of these poles in the complex plane for $K \geq 0$. Indicate the pole locations for $K = 0, 1, 2,$ and 3 on the locus.



$T_1(s)$ Transfer function for inner loop.

$$T_1 = \frac{\frac{1}{s}}{1 + \frac{1}{s}\cdot 2K} = \frac{1}{s+2K}$$

Elements in series

$$T_2 = \frac{1}{s(s+2K)}$$

Total transfer function including outer loop.

$$T = \frac{T_2}{1 + T_2 \cdot 4} = \frac{1}{s(s+2K)+4} = \frac{1}{s^2+2Ks+4}$$

b) Poles

$$s = \frac{-2K \pm \sqrt{4K^2-16}}{2} = -K \pm j\sqrt{4-K^2}$$

K acts like a damping element.

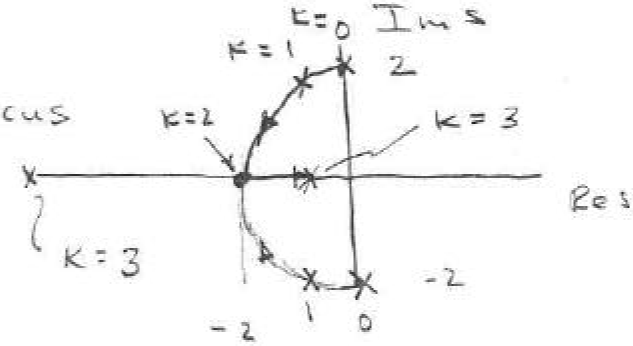
$K = 0 \qquad s = \pm 2j$

$K = 1 \qquad s = -1 \pm \sqrt{3}\, j$

$K = 2 \qquad s = -2$ (critically damped)

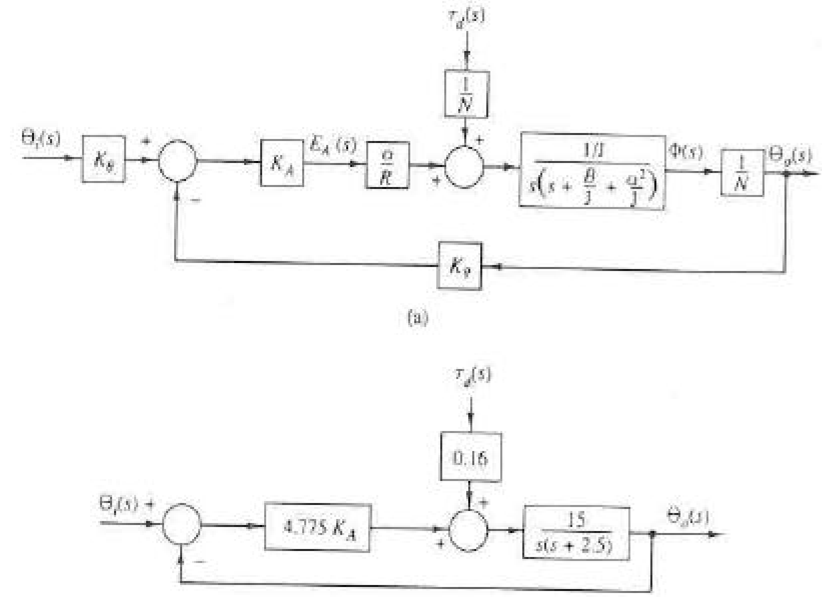
$K = 3 \qquad s = -3 \pm \sqrt{5} \Rightarrow s = -5.24, -0.76$

Root locus



* **14.28** The servomechanism discussed in Section 14.1 has a disturbance torque $\tau_d(t)$ applied to the shaft to which the output potentiometer is attached. In Figure 14.1, the positive sense of this torque is clockwise.

a) Show that the block diagram of Figure 14.2 must be modified to appear as in Figure P14.28(a).



(b)

FIGURE P14.28

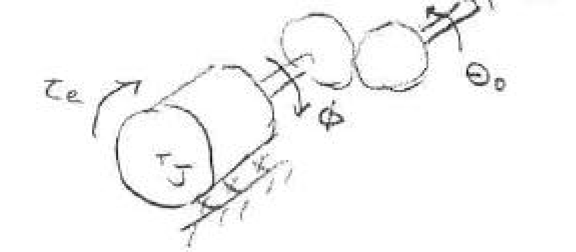
b) When $R = 8\ \Omega$, $\alpha = 5.0$ V·s/rad, and the two potentiometer gain blocks K_θ are moved to the output of the summing junction and combined with the amplifier gain block K_A, verify that the model can be represented as shown in Figure P14.28(b). Use the numerical values in Table 14.1.

c) Assume that proportional control is used with $K_A = 0.08727$ V/V. Determine the steady-state errors to

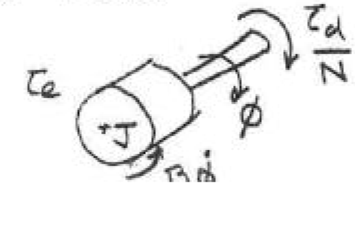
(i) A unit ramp input in the reference angle $\theta_i(t)$.

(ii) A unit step in the disturbance torque $\tau_d(t)$.

Consider the free body diagram for the armature/shaft (refer to figure 14.1.)



The effect of the gears is to modify the disturbance torque by the gear ratio



Thus, the equation of motion is

$$J\ddot{\phi} + B\phi = \tau_e + \frac{\tau_d}{N} \qquad (1)$$

But $\tau_e = \alpha i_A = \frac{\alpha}{R}\left(e_a - \alpha\dot{\phi}\right)$ (2) from (4b) p 527

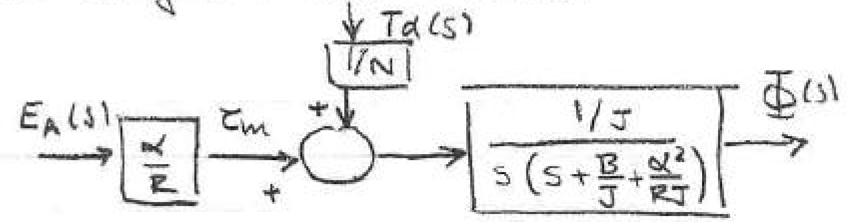
Take the Laplace transform of (1) & (2) with zero initial conditions $\Rightarrow$

$$\left[Js^2 + \left(B + \frac{\alpha^2}{R}\right)s \right]\Phi(s) = \frac{\alpha}{R}E_a + \frac{\tau_d(s)}{N}$$

or

$$\Phi(s) = \left(\frac{\alpha}{R}E_a(s) + \frac{\tau_d(s)}{N}\right) \cdot \frac{1/J}{s^2 + \left(\frac{B}{J} + \frac{\alpha^2}{RJ}\right)s}$$

This relationship can be represented using the block diagram shown below



This is consistent with the diagram shown except for an error in the book. The book left out the "R" in the term $\frac{\alpha^2}{RJ}$.

b). Block becomes

$$K_\Theta K_A \frac{\alpha}{R} = \frac{15}{\pi} \cdot \frac{5}{8} K_A =$$

b). From Table 14.1

$A = 15 \Rightarrow K_\theta = \frac{15}{\pi} = 4.775$

$K_m = 150$

$\tau_m = .4$

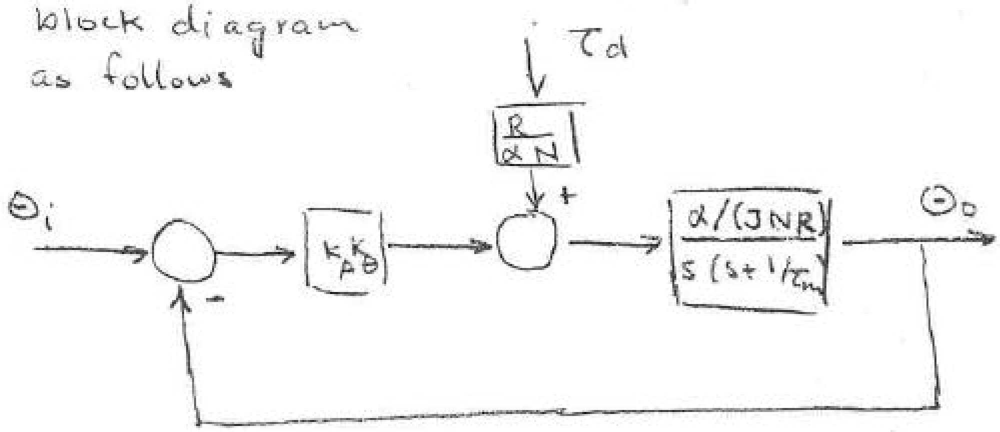
$N = 10.$

Combine $K_m = \frac{\alpha}{RJ}$ and $\frac{1}{N}$

$$\Rightarrow \boxed{\frac{\alpha/JR}{s(s+1/\tau_m)}} \longrightarrow \boxed{1/N} \xrightarrow{\Theta_o} \;\Longrightarrow\; \longrightarrow \boxed{\frac{150 \cdot \frac{1}{10}}{s(s+1/.4)}} \xrightarrow{\Theta_o}$$

$$\Rightarrow \longrightarrow \boxed{\frac{15}{s(s+2.5)}} \xrightarrow{\Theta_o}$$

Thus the loop can be replaced with an equivalent block diagram as follows



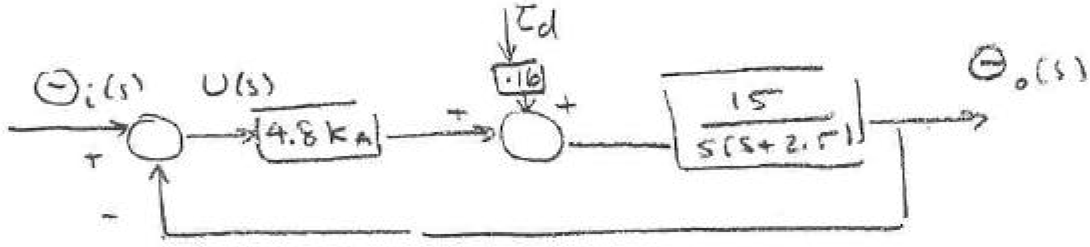
Thus
$$\frac{\frac{\alpha}{JNR}}{s(s+1/\tau_m)} = \frac{15}{s(s+2.5)}$$

$$\frac{R}{\alpha N} = \frac{8}{5 \cdot 10} = 0.16$$

$$K_A K_\theta = 4.775\, K_A$$

This gives figure P14.28 (b).

c) The block diagram is



From the diagram

$$\Theta_o(s) = \frac{15}{s(s+2.5)} \cdot \left(0.16\,\tau_d + 4.8K_A\, U(s)\right)$$

But $U(s) = \Theta_i - \Theta_o$

$$\Rightarrow \quad \Theta_o(s) = \frac{15}{s(s+2.5)}\left[0.16\,\tau_d + 4.8K_A\left(\Theta_i - \Theta_o\right)\right]$$

$$\Rightarrow \quad \Theta_o = \frac{15}{s(s+2.5)} \frac{\left(0.16\,\tau_d + 4.8K_A\Theta_i\right)}{\left(1 + \frac{15 \cdot 4.8K_A}{s(s+2.5)}\right)}$$

or $$\Theta_o = \frac{15\left[0.16\,\tau_d + 4.8K_A\,\Theta_i(s)\right]}{s^2 + 2.5s + 15 \cdot 4.8 \cdot K_A}$$

i) $\Theta_i(t) = t, \quad \tau_d = 0$

$$\Theta_i(s) = \frac{1}{s^2}$$

$$\Theta_o = \frac{15 \cdot 4.8K_A \cdot 1/s^2}{s^2 + 2.5s + 15 \cdot 4.8 \cdot K_A}$$

$$\Theta_o(s) = \frac{A}{s^2} + F(s)$$

$$A = \lim_{s \to 0} s^2\, \Theta_o(s) = 1$$

$$F(s) = \frac{15 \cdot \cancel{4}.8 K_A - (s^2 + 2.5s + 15 \cdot \cancel{4}.8 K_A)}{s^2 (s^2 + 2.5s + 15 \cdot 4.8 K_A)}$$

$$= \frac{1}{s} \; \frac{-s - 2.5}{s^2 + 2.5s + 15 \cdot 4.8 K_A}$$

$F(s) \to$ step function + damped oscillation

$\therefore$ Long time behavior given by $\frac{1}{s^2} \to t$

There is no steady state error.

ii) Suppose $T_d(t) = H(t)$ and $\Theta_i = 0$

$$\Theta_o(s) = \frac{15 \cdot 0.16 \cdot 1/s}{s^2 + 2.5s + 15 \cdot 4.8 \cdot K_A}$$

Final value theorem $\Rightarrow$

$$\Theta_o(\infty) = \lim_{s \to 0} s\, \Theta_o(s) = \frac{\cancel{15} \cdot 0.16}{\cancel{15} \cdot 4.8 K_A}$$

$$\Theta_o(\infty) = \frac{0.033}{K_A}$$