8. PDE

8.1 Introduction

Examples of PDE, u is a function of x and y

	order?	linear or non-linear?
$\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y^2} + u = 1$		
$\frac{\partial^3 u}{\partial x^2 \partial y} + x\frac{\partial^2 u}{\partial y^2} + 8u = 5y$		
$\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + 6\frac{\partial^3 u}{\partial x \partial y^2} = x$		
$\frac{\partial^2 u}{\partial x^2} + xu\frac{\partial u}{\partial y} = y$		

We will focus on linear, second-order PDEs, because they have most widely spread applications in engineering.

For two indep. variables such PDEs can be expressed in the following general form:

$$\boxed{\phantom{Au_{xx} + Bu_{xy} + Cu_{yy} + D = 0}} \quad — (*)$$

A, B, C :

D :

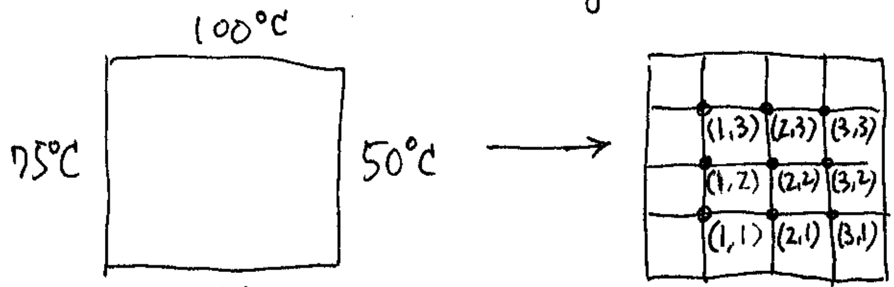
Depending on the values of the coefficients of the second-derivative terms, A, B and C, (*) can be classified into one of three categories.

B^2-4AC	Category	Examples.
<0	Elliptic	$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$ (Laplace eq) $=f(x,y)$ (Poisson's eq)
$=0$	Parabolic	$\frac{\partial T}{\partial t}=k\frac{\partial^2 T}{\partial x^2}$ (Heat conduction eq)
>0	Hyperbolic	$\frac{\partial^2 y}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$ (Wave eq)

Note: For cases where A, B and C depend on x and y, the equation may fall into a different category depending on the location in the domain.

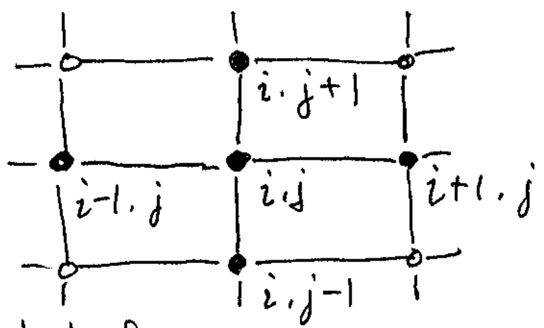
8.2 Finite Difference Solution to Elliptic Eq.

Laplace eq: $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0$



PDE $\longrightarrow$ Algebraic finite difference equation.

The Laplacian Difference Eq.



centered f-d-d formulas:

$$\frac{\partial^2 T}{\partial x^2} = \underline{\qquad\qquad\qquad\qquad} + O(\quad)$$

$$\frac{\partial^2 T}{\partial y^2} = \underline{\qquad\qquad\qquad\qquad} + O(\quad)$$

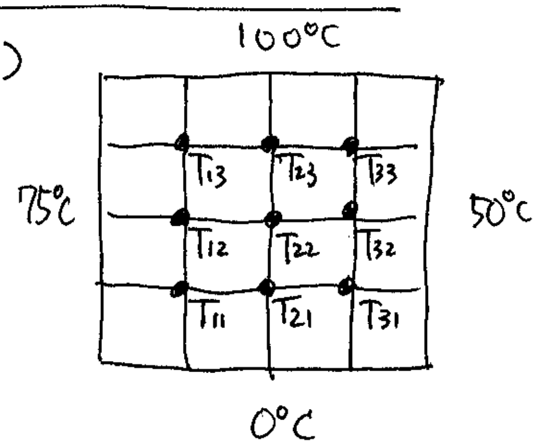
Plug the two f-d-d formulas into the Laplace eq. For a square grid $\Delta x = \Delta y$ and by collecting terms we get:

$$\boxed{\qquad\qquad\qquad\qquad\qquad\qquad}$$

In addition to the Laplacian difference eq.
boundary conditions (BC) along the edges of the plate must be specified to obtain a unique solution.

$$\text{BC} \begin{cases} \text{Dirichlet BC} \\ \text{Neumann BC} \end{cases}$$

Ex. 4x4 grid (5x5 grid point) heat transfer problem



Laplacian diff. eq. for node (1,1)

$$T + T + T + T - 4T = 0$$

with Dirichlet BC

$$-4T_{11} + T_{12} + T_{21} = -75$$

$$\left[\begin{array}{c|c|c} \quad & \quad & \quad \\ \hline \quad & \quad & \quad \\ \hline \quad & \quad & \quad \end{array}\right] \begin{Bmatrix} T_{11} \\ T_{21} \\ T_{31} \\ T_{12} \\ T_{22} \\ T_{32} \\ T_{13} \\ T_{23} \\ T_{33} \end{Bmatrix} = \begin{Bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{Bmatrix}$$

Which method would you use to solve the eq?

Secondary variables

Because its distribution is described by the Laplace equation, $\boxed{}$ is considered to be the primary variable in the heated-plate problem.

The secondary variable is the derivative of the primary var, $\boxed{}$

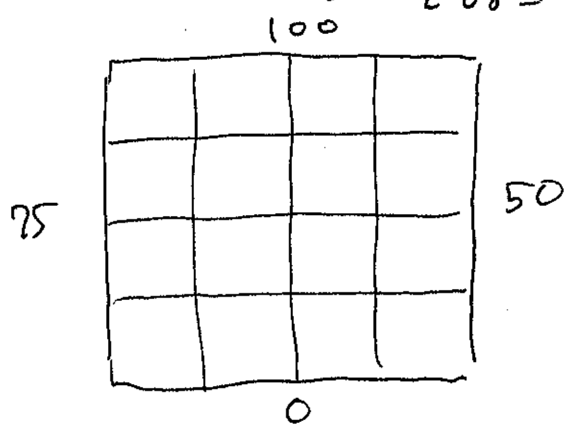
Central f-d-d approximation for the first deriv.

$$q_x = -k\rho C\,\frac{\partial T}{\partial x} =$$

$$q_y = -\underbrace{k}_{\substack{\text{coeff. of}\\ \text{thermal}\\ \text{diffusivity}\\ (cm^2/s)}}\underbrace{\rho}_{\substack{\text{density}\\ (g/cm^3)}}\underbrace{C}_{\substack{\text{heat capacity}\\ (cal/(g\cdot{}^\circ C))}}\frac{\partial T}{\partial y} =$$

what is the truncation error? $O(\quad)$

Heat flux vector : $\vec{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$

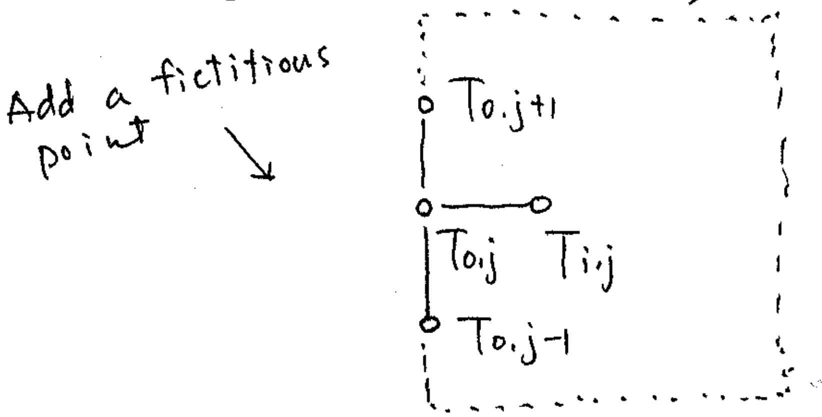


(*) For PS12 (4), use $k\rho C = 1$.

Neumann BC

A common alternative to Dirichlet BC is the case where the derivative is given.
In the heated-plate problem this is to specify the heat flux.

One example of Neumann BC is the situation where the edge is insulated, or the heat flux is zero.
(This is called natural BC.)



Laplacian diff. eq. for $T_{0,j}$

$$\boxed{} \quad \text{—(1)}$$

$$\frac{\partial T}{\partial x} = \underline{} \quad \text{—(2)}$$

$$T_{-1,j} = T_{1,j} - 2\Delta x \frac{\partial T}{\partial x} \quad \text{—(3)}$$

(3) → (1)

$$\boxed{}$$