7. ORDINARY DIFFERENTIAL EQUATION

7.1 Introduction

DE ┬ ODE   # of indep. var = 1
   └ PDE   # of indep. var ≥ 2

[dependent variable
[independent variable

[linear ODE   ← we will deal with this type only in this class.
[non linear ODE

Def. of linear ODE

$$\underbrace{a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y}_{\substack{\text{linear combination of} \\ y,\ y',\ y'',\ y''',\ \cdots y^{(n)}}} = F(x)$$

Question : which ones are linear ODE?

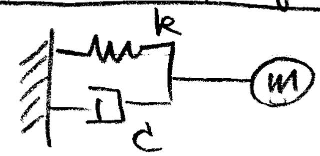
① $\frac{dy}{dx} = x+1$ ② $\frac{dy}{dx} = y+1$

③ $\frac{dy}{dx} = x^3+1$ ④ $\frac{dy}{dx} = (x^3+1)y$

⑤ $\frac{dy}{dx} = y^2+1$ ⑥ $\frac{dy}{dx} = \sin y$

ANSWER : ①,②,③,④

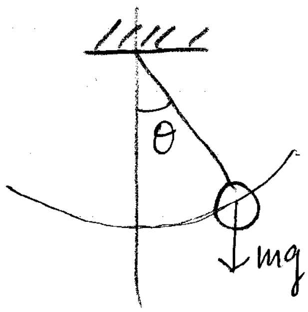
e.g.) Mass-spring-damper system



$\leftrightarrow F(t) = C\sin\omega t$

$m\ddot{x} + c\dot{x} + kx = F(t)$ ← linear

e.g.) Swinging Pendulum



$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

↑

nonlinear, but if $\theta \simeq 0$

$$\boxed{\sin\theta = \theta} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

linearized version.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

$\boxed{\text{Higher order linear ODE}}$

$\downarrow$

$\boxed{\text{Coupled 1st order ODEs}}$

This conversion is necessary in order apply Euler's / Runge-Kutta methods to solving a higher order ODE.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = F(x) \Leftarrow$$ nth order ODE

$$\downarrow$$

$$y_0 = y$$
$$y_1 = y'$$
$$y_2 = y''$$
$$\vdots$$
$$y_{n-1} = y^{(n-1)}$$

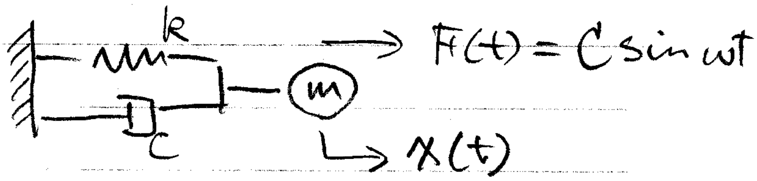
$$\downarrow$$

$$\begin{cases} y_0' = y_1 \\ y_1' = y_2 \\ y_2' = y_3 \\ \vdots \\ y_{n-1}' = \dfrac{f(t) - a_0 y_0 - a_1 y_1 - \cdots - a_{n-1} y_{n-1}}{a_n}. \end{cases} \Leftarrow$$ Coupled 1st order ODEs.

$\boxed{\text{Vector form} \quad y' = f(x, y)}$

indep. var. ↑ ↑ dep. var.

e.g.) 2nd order ODE → 1st order ODE



indep var : t

dep var : $x(t)$

original 2nd order ODE.

$$m x'' + c x' + k x = F(t)$$

$$\downarrow$$

$$\begin{cases} y_0 = x \\ y_1 = x' \end{cases} \qquad y = \begin{Bmatrix} y_0 \\ y_1 \end{Bmatrix}$$

state variables : y_0, y_1

$$\downarrow$$

$$\begin{cases} y_0' = y_1 \\ y_1' = \dfrac{F(t) - k y_0 - c y_1}{m} \end{cases}$$

$$\downarrow$$

$$\boxed{y' = f(t, y)}$$

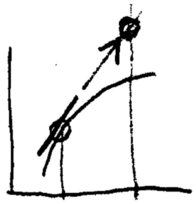
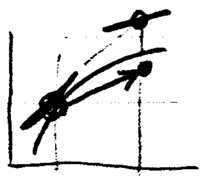
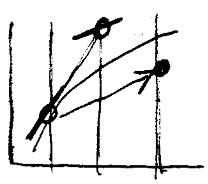
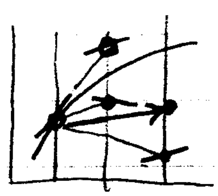
state variable

$$y = \begin{Bmatrix} y_0 \\ y_1 \end{Bmatrix} = \begin{Bmatrix} x \\ x_1 \end{Bmatrix}$$

this function is also a vector

$$f = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ \dfrac{F(t) - k y_0 - c y_1}{m} \end{Bmatrix}$$

ODE Integration Scheme Summary

		global error	exact for	Runge-Kutta
Euler's		$O(h)$	linear func.	1st order Runge-Kutta
Heun's		$O(h^2)$	quadratic	2nd order Runge-Kutta
Mid point		$O(h^2)$	quadratic	2nd order Runge-Kutta
Classical Runge-Kutta		$O(h^2)$	quartic	4th order Runge-Kutta

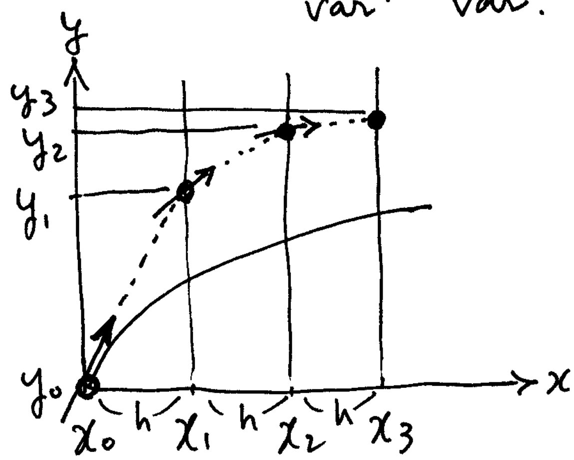
most popular RK method for practical applications

7.3 Euler's Method

Input $\boxed{y' = f(x, y)}$

vectors (y', f)   indep var (x)   dep var. (y)

The 1st order derivative (or slope) is a function of both x and y.



Euler's method (1st order Taylor approx)

$$\boxed{y_{i+1} = y_i + f(x_i, y_i)h} + O(h^2)$$

<u>local</u> truncation error in one step, $x_i \to x_{i+1}$

<u>global</u> truncation error (always greater than local)

$$\# \text{ of steps} \times \underset{\text{(local)}}{O(h^2)}$$

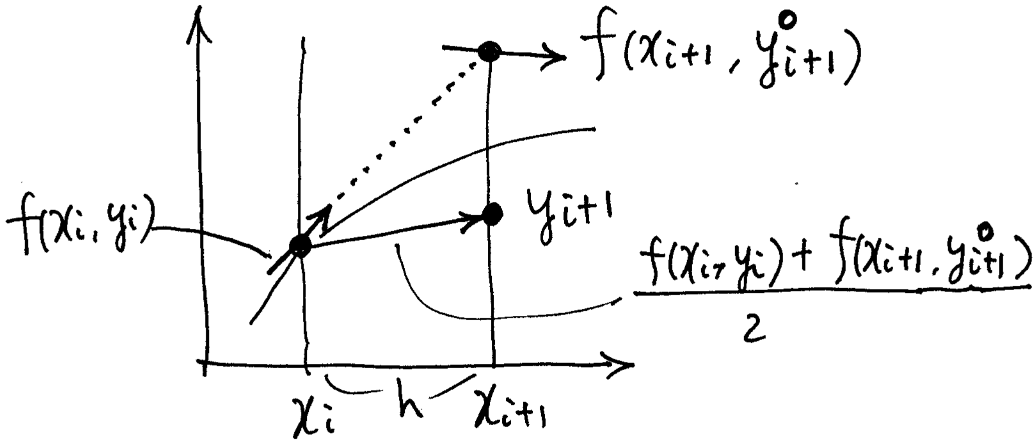
$$= O\left(\tfrac{1}{h}\right) \times O(h^2)$$

$$= O(h)$$

$O(h^2)$ is better than $O(h)$

Heun's Method

(predictor – corrector approach)



Heun's method

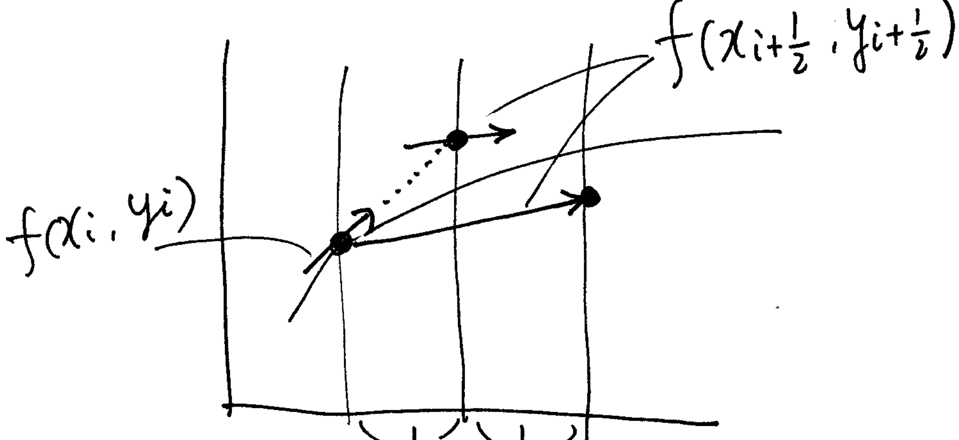
$$y_{i+1}^0 = y_i + f(x_i, y_i)h$$

$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)}{2} h$$

corrector

predictor

Midpoint Method



Midpoint

$$\boxed{\begin{aligned} y_{i+\frac{1}{2}} &= y_i + f(x_i, y_i)\cdot \frac{h}{2} \\ y_{i+1} &= y_i + f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})\cdot h \end{aligned}}$$

$$x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$$

(*) Both Heun's and midpoint methods are examples of the 2nd order Runge-Kutta method.

7.3 Runge-Kutta Methods

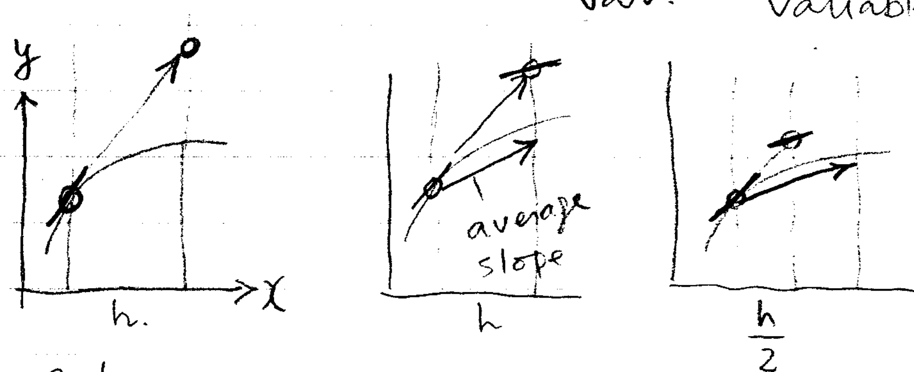
higher order **linear** ODE $\rightarrow$ $\boxed{y' = f(x, y)}$ $\leftarrow$ y & f are vectors.

indep. var. (x) dependent variable (y)

make sure you understand what 'linear' means.

$\frac{dy}{dx} = x^2 + y$ (linear)

$\frac{dy}{dx} = y^2 + x$ (non-linear)



Euler. Heun. Mid-Point

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{examples of R-K methods}}$

examples of R-K methods. (one step methods)

- weighted average
- sub-steps

to estimate a representative slope more accurately

<u>Generalized form of RK solutions.</u>

$$y_{i+1} = y_i + \underbrace{\phi(x_i, y_i, h)}_{\text{Reprentative slope}}\, h.$$

where $\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$

RK-1st

RK-2nd $\quad k_1 = f(x_i, y_i)$

$\quad k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$

RK-3rd

$\quad k_3 = f(x_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h)$

RK-4th

$$k_4 = f(x_i + p_3 h, y_i + q_{31} k_1 h + q_{32} k_2 h + q_{33} k_3 h)$$

(*) Note:

k's are recurrence relationships.
That is, k_1 appears in the eq
for k_2, which appears in the eq
for k_2, and so forth

RK-1st

$\longrightarrow$ Euler.

RK-2nd

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) h \quad \text{---} (*)$$

where $k_1 = f(x_i, y_i)$

$k_2 = f(x_i + p_1 h,\ y_i + q_{11} k_1 h)$

4 unknowns : a_1, a_2, p_1, q_{11}

read textbook p.697 Box 25.1 → use algebraic manipulation to solve for values of unknown that make eq $(*)$ equivalent to the 2nd order Taylor series approx.

$$\begin{cases} a_1 + a_2 = 1 \\ a_2 p_1 = 1/2 \\ a_2 q_{11} = 1/2 \end{cases}$$

⇐ typo in the textbook (3rd Edition). $\cancel{a_1 p_2} = 1/2$

also (25.32) on p.696 is a typo.

$$\begin{cases} a_1 = 1 - a_2 \\ p_1 = 1/2a_2 \\ q_{11} = 1/2a_2 \end{cases}$$

this is why the 2nd order RK is exact to quadratic func.

The common basic strategy underlying all the Runge-Kutta methods

Three simultaneous eqs for four unknowns
(one more unknown than the # of eqs)
→ no unique set of solutions.
→ by assuming a value for one
we can determine the other three

$a_2 = 1/2$ (Heun's)

$a_1 = 1/2, \quad p_1 = q_{11} = 1$

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where $k_1 = f(x_i, y_i)$

$k_2 = f(x_i + h, y_i + k_1 h)$

$a_2 = 1$ (Mid point)

$a_1 = 0, \quad p_1 = q_{11} = 1/2$

$$y_{i+1} = y_i + k_2 h$$

where $k_1 = f(x_i, y_i)$

$k_2 = f(x_i + \frac{h}{2}, y_i + k_1 \frac{h}{2})$

$a_2 = 2/3$ (Ralston) ← minimum trunc. error

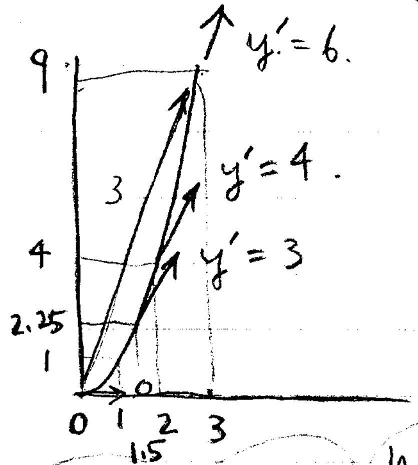
$a_1 = 1/3, \quad p_1 = q_{11} = 3/4$

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

where $k_1 = f(x_i, y_i)$

$k_2 = f(x_i + \frac{3}{4}h, y_i + k_1 \frac{3}{4}h)$

ex.) $\overset{f(x,y)}{y' = 2x} \longrightarrow y = x^2$.



exact solution to the slope

$\phi = 3$

$x_i = 0 \quad x_{i+1} = 3. \quad h = 3.$

Heun's $\quad k_1 = f(0,0) = 0$

$k_2 = f(3,0) = 6$

$\phi = \frac{1}{2}k_1 + \frac{1}{2}k_2 = 3$ //

Midpoint $\quad k_1 = f(0,0) = 0$

$k_2 = f(\frac{3}{2}, 0) = 3$.

$\phi = k_2 = 3$ //

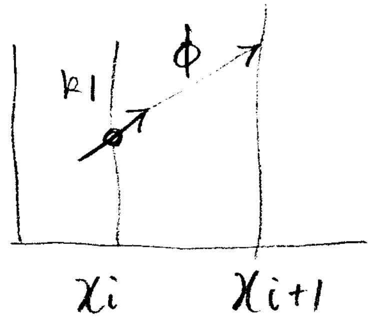
Ralston

$k_1 = f(0,0) = 0$

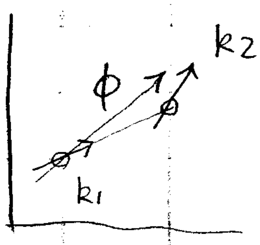
$k_2 = f(\frac{9}{4}, 0) = \frac{9}{2}$

$\phi = \frac{1}{3}k_1 + \frac{2}{3}k_2 = \frac{2}{3} \cdot \frac{9}{2} = 3$ //

RK-1st

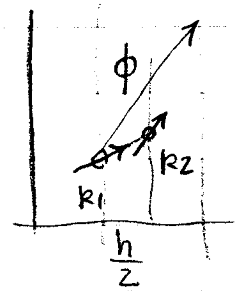


RK-2nd



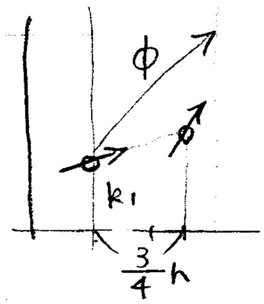
Henn

$a_1 = a_2 = 1/2$



Mid-point

$a_1 = 0, \ a_2 = 1$



Ralston

$a_1 = 1/3, \ a_2 = 2/3$

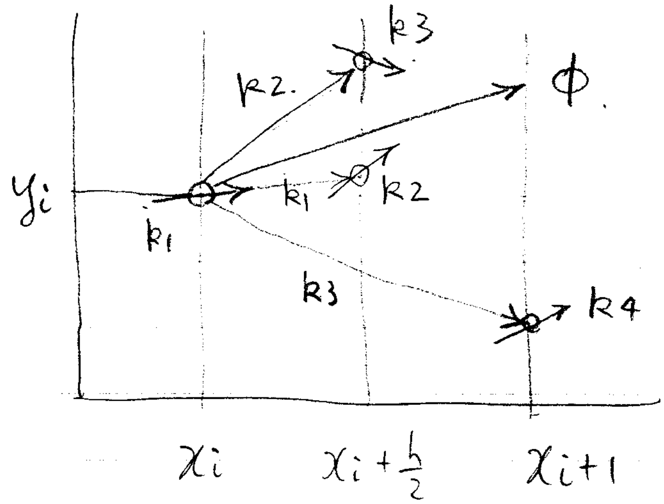
$\boxed{\text{RK-4th}}$

most popular RK methods are 4th order

$$y_{i+1} = y_i + \frac{k_1 + 2k_1 + 2k_3 + k_4}{6} h$$

where
$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{h}{2},\ y_i + k_1 \frac{h}{2}\right)$$
$$k_3 = f\left(x_i + \frac{h}{2},\ y_i + k_2 \frac{h}{2}\right)$$
$$k_4 = f(x_i + h,\ y_i + k_3 h)$$

$$a_1 = \frac{1}{6},\quad a_2 = \frac{1}{3},\quad a_3 = \frac{1}{3},\quad a_4 = \frac{1}{6}$$



(*) this formula is equivalent to the 4 th order taylor series approximation
local error (one step) $O(h^5)$
global error ($\frac{1}{h}$ steps) $O(h^4)$
exact to the quartic-function