Stresses on Inclined Planes in 3-D

You have learned about plane stress and the transformation of stresses. In that case we had only three stress components that acted in a single plane and we took them to be σ_x, σ_y, and τ_{xy}. These stresses are shown in Figure 1 acting on an element that is drawn in a 2-D fashion and acting on a 3-D element.

We considered the normal and shear stresses, σ and τ, acting on planes inclined with respect to the x and y-axes. That is, the normal vectors to these planes all lay in the x-y plane. The triangular element from which one can determine the transformation formulas for $\sigma(\theta)$ and $\tau(\theta)$ using equilibrium conditions are shown in Figure 2. Two other elements with surfaces inclined at different angles are shown in Figure 3, drawn both in 2-D and 3-D.
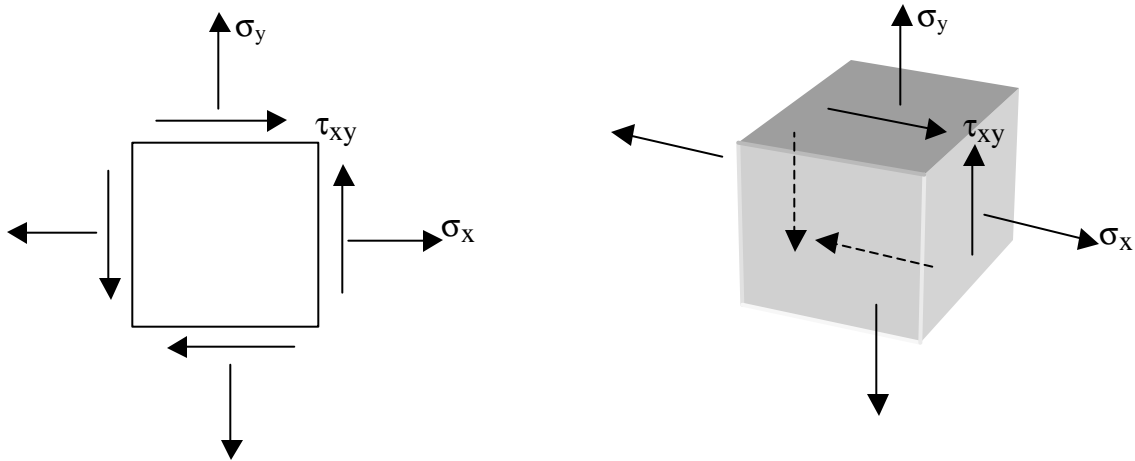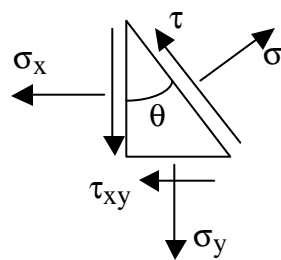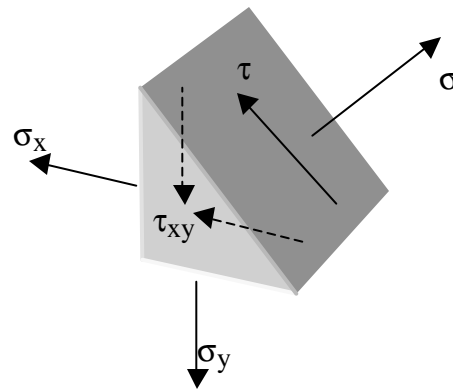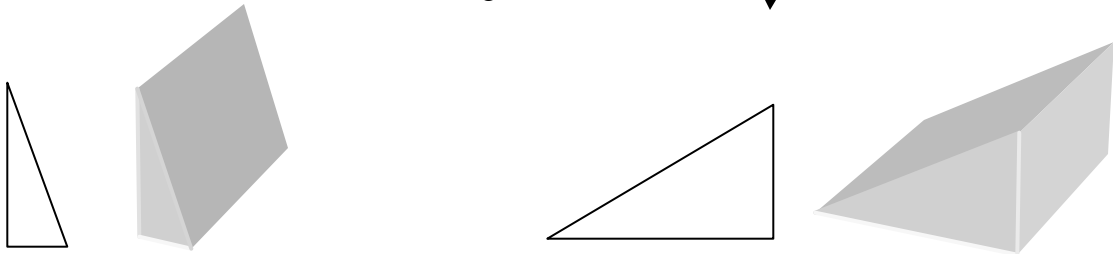
Figure 1

Figure 2

Figure 3

The transformation formulas are:

$$s(q) = \frac{1}{2}(s_x + s_y) + \frac{1}{2}(s_x - s_y)\cos(2q) + t_{xy}\sin(2q)$$

$$t(q) = t_{xy}\cos(2q) + \frac{1}{2}(s_y - s_x)\sin(2q)$$

The important quantities that are derived from the transformation formulas are:

$$t_{max} = \sqrt{\frac{1}{4}(s_y - s_x)^2 + t_{xy}^2}$$

$$q_s = \frac{1}{2}\tan^{-1}\left[\frac{(s_y - s_x)}{2t_{xy}}\right]$$

$$s_m = \frac{1}{2}(s_x + s_y)$$

The maximum shear stress (from among all angles θ) has magnitude τ_{max}. This shear stress acts on planes inclined at angles of θ_s, $\theta_s +90°$, $\theta_s +180°$, and $\theta_s +270°$. To know the precise directions of the shear stress on these planes, one should evaluate the formula $\tau(\theta)$ at one of these planes. The normal stress on all those four planes is equal to σ_m; these normal stress may be positive, negative or zero. This square with four faces inclined at θ_s, $\theta_s +90°$, $\theta_s +180°$, and $\theta_s +270°$ is called the Maximum Shear Stress Element.

The normal stress has maximum and minimum values (from among all angles θ), which are denoted by σ_{max} and σ_{min}, or σ_1 and σ_2; these are called principal stresses. These normal stresses act on planes oriented at 45 degrees away from the planes of the Maximum Shear Stress Element; namely at $\theta_s +45°$, $\theta_s +135°$, $\theta_s +225°$, and $\theta_s +315°$. The directions perpendicular to these planes are called principal axes or principal directions. The principal stresses can be found from by $\sigma_{max} = \sigma_1 = \sigma_m + \tau_{max}$ and $\sigma_{min} = \sigma_2 = \sigma_m - \tau_{max}$. One can also write $\tau_{max} = (\sigma_1 - \sigma_2)/2$. The shear stresses on the principal axes are zero. The square with faces inclined at θ_s, $\theta_s +90°$, $\theta_s +180°$, and $\theta_s +270°$ is referred to as the Principal Stress Element
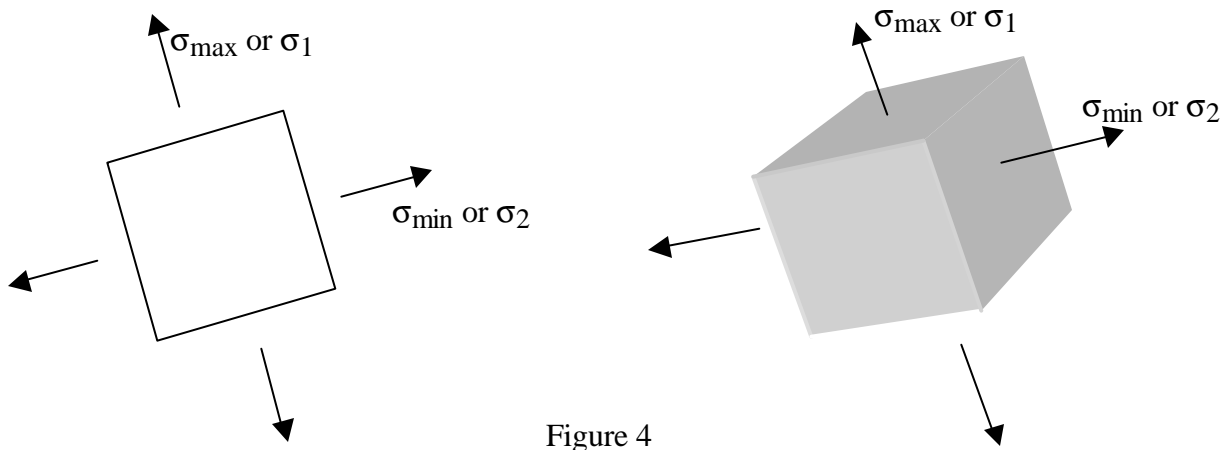
Figure 4

When all stresses components σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, and τ_{zy}, are present (not zero), it is still true that one can always find three perpendicular planes on which the shear stresses are zero. See Figure 5. (This requires advanced mathematics to prove). The normal stresses on these planes, σ_1, σ_2, σ_3, are again called principal stresses and the directions normal to these planes are called principal directions or axes. One of these three stresses is the maximum of all normal stresses and one is the minimum of all normal stresses. Our planar case above just corresponded to the situation in which one of the three principal stresses is zero.
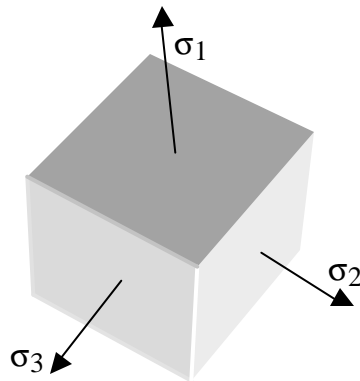

Figure 5

One can find the shear stress on the three planes that lie at 45° to each pair of principal axes. In each case, one can isolate the triangular prism above or below one plane at 45°. Such triangular prisms are just what we used to get the transformation formulae above, as in Figures 2 and 3). Consider, for example, the left cube in Figure 6. The stress σ_3 acts parallel to the axis of the prism or perpendicular to the triangular face; there are equal and opposite forces contributed by σ_3 acting on the two end faces of the prism. σ_3 acts perpendicularly to the shear stress on the inclined face in which we are interested and so has no effect on this shear stress. Therefore, the determination of this shear stress reduces to the planar case above. Without worrying about its direction or sign, the shear stress on this 45° plane is $\tau = (\sigma_1 - \sigma_2)/2$. One can likewise find the other shear stresses as shown in Figure 6. Note that the sign of the shear stress is not determined by these equations.

$$\tau = (\sigma_1 - \sigma_2)/2 \qquad \tau = (\sigma_1 - \sigma_3)/2 \qquad \tau = (\sigma_2 - \sigma_3)/2$$
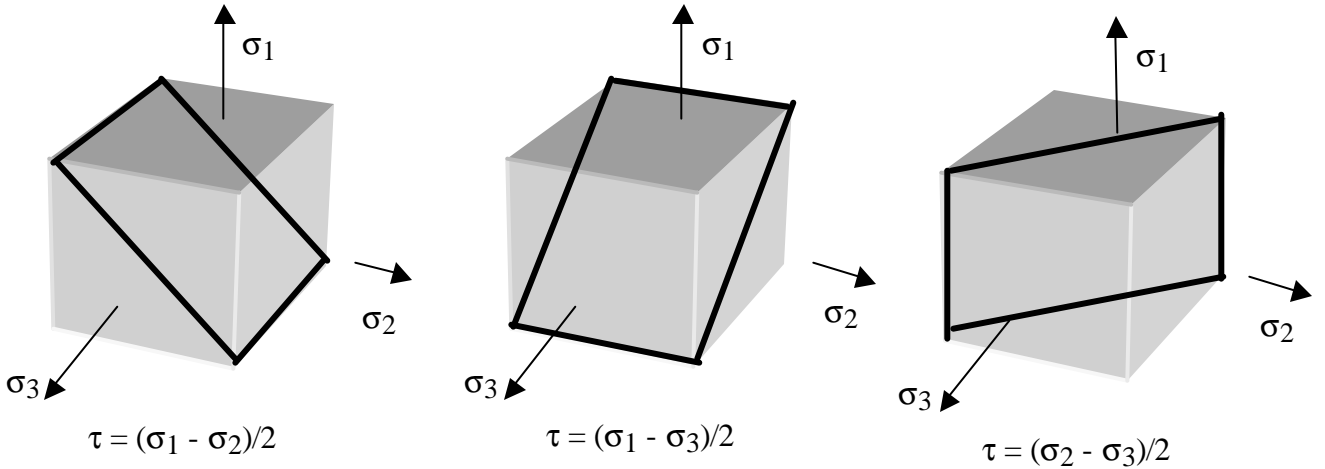
Figure 6

It can be proved (again the mathematics is more advanced) that the largest of these three shear stresses is the largest shear stress on *ANY* plane in the cube, no matter how it is oriented. If one defines the numbering 1, 2, 3 of the three principal stresses so that they are ordered according to $\sigma_1 > \sigma_2 > \sigma_3$, then the absolute maximum shear stress is $\tau_{abs} = (\sigma_1 - \sigma_3)/2$.

Returning to the planar case, which is very common, we will now show that the maximum in-plane shear stress τ_{max}, which we found from the transformation formulas above, may or may not be the absolute maximum shear stress. Remember that τ_{max} was defined in terms of the difference between the two *in-plane* principal stresses. Whereas we called them σ_1 and σ_2 before, that was before we considered the possibility of the third principal stress being non-zero.

Consider the three cases shown in Figure 7. In each case recall that the third principal stress is equal to zero. That third principal stress could be denoted by σ_1 or σ_2 or σ_3, depending on the values of the other two. The absolute maximum shear stress is τ_{abs} is equal to one-half of the difference between the maximum and minimum principal stresses.
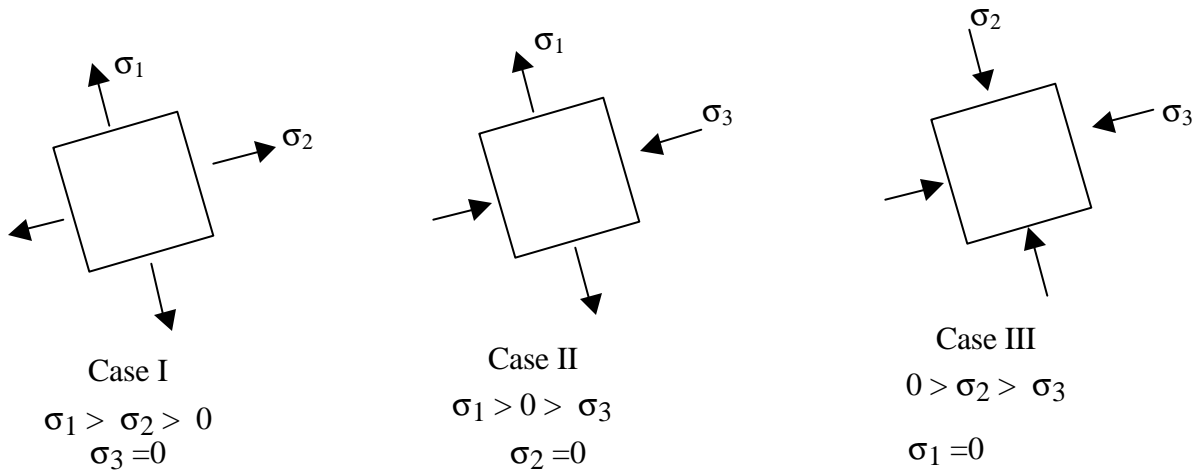
Figure 7

For case I, the maximum and minimum principal stresses are σ_1 and σ_3, respectively, and so the absolute maximum shear stress $(\sigma_1 - \sigma_3)/2$ is greater than the in-plane maximum shear stress $(\sigma_1 - \sigma_2)/2$. For case II, the maximum and minimum principal stresses are σ_1 and σ_3, respectively, and so the absolute maximum shear stress is equal to the in-plane maximum shear stress (which are both $(\sigma_1 - \sigma_3)/2$). For case III, the maximum and minimum principal stresses are σ_1 and σ_3, respectively, and so the absolute maximum shear stress $(\sigma_1 - \sigma_3)/2$ is greater than the in-plane maximum shear stress $(\sigma_2 - \sigma_3)/2$. Case II, in which one in-plane principal is positive and one is negative arises in the common situation of one normal stress and one shear stress (say, σ_x and τ_{xy}) such as occurs under a combination of tension and torsion. Case I could arise, for example in a pressure vessel.

Failure Criteria

Typically, we obtain data on failure of a material under excessive stress by testing the material in uniaxial tension. Under a general loading situation, at a single point there may be several stress components acting, including both normal and shear stresses. How does one determine whether failure occurs under this combination of stress components? A failure criterion provides a means of judging whether the combination of stress would cause failure.

We consider two idealized types of material failure: ductile and brittle.

Ductile Failure

For ductile materials such as metals and many plastics, failure is said to occur when there is plastic or permanent deformation, that is when the tensile stress-strain curve deviates from linearity. It has been determined that such plastic deformation occurs under the action of shear stress. There are two primary means of combining stresses so as to predict whether plastic deformation will occur.

Maximum Shear Stress Yield Criterion

One considers all the stresses and determines the absolute maximum shear stress. This shear stress is compared with the shear stress which, if acting alone, would cause yielding or plastic deformation; we term this shear stress at yielding τ_Y. However, mechanical testing usually occurs in <u>uniaxial</u> tension. At any level of tensile stress, we have learned that the absolute maximum shear stress is equal to one half of the tensile stress. Therefore, if the yield stress in uniaxial tension is σ_Y, then τ_Y is given by $\tau_Y = \sigma_Y / 2$.

Mises Yield Criterion

A second yield criterion is based on capturing the average level of shearing rather than the absolute maximum.

We define the Mises or equivalent stress, σ_e, by the following equation:

$$\boldsymbol{s}_e = \sqrt{\frac{1}{2}\left[(\boldsymbol{s}_1 - \boldsymbol{s}_2)^2 + (\boldsymbol{s}_1 - \boldsymbol{s}_3)^2 + (\boldsymbol{s}_2 - \boldsymbol{s}_3)^2\right]}$$

Notice that this is some type of average of the shear stresses. It is proportional to a root mean square of the three shear stresses that were found on planes at 45° from the principal planes.

Consider the special case when there is only uniaxial tension σ. Then, $\sigma_1 = \sigma$, and $\sigma_2 = \sigma_3 = 0$. Substitute into the formula for the Mises stress and you find that $\sigma_e = \sigma$.

The Mises Yield criterion states that a ductile material will yield when the Mises or equivalent stress, σ_e reaches a critical value. Since $\sigma_e = \sigma$ for the particular case of uniaxial tension, and since we have denoted the yield stress in uniaxial tension by σ_Y, then, we can see that yielding occurs whenever σ_e reaches σ_Y, regardless of the particular stress components which leads to the value of σ_e.

The Mises Yield criterion is convenient for the following reason. We actually don't have to find the principal stresses σ_1, σ_2, and σ_3 to calculate σ_e. Using some additional advanced mathematics, it can be shown that σ_e is also related to the full set of components by

$$s_e = \sqrt{\frac{1}{2}\left[(s_x - s_y)^2 + (s_x - s_z)^2 + (s_y - s_z)^2 + 6t_{xy}^2 + 6t_{xz}^2 + 6t_{yz}^2\right]}$$

Therefore, one can bypass the calculation of the principal stresses (which is difficult), and immediately find the Mises stress σ_e and apply the Mises Yield criterion. Finite element programs typically compute the Mises stress σ_e at nodal points.

Brittle Failure

For brittle materials, such as ceramics, glasses, concrete, and cast iron, failure tends to occur more catastrophically with the material cracking suddenly. This is idealized to occur when the tensile stress reaches a critical value. One would find this critical level of tensile stress from a uniaxial tension test, or more likely from a bend test.

Therefore, given any set of stresses, one determines the maximum normal stress from among all possible planes and compares this stress with the critical stress for failure. This maximum normal stress is the maximum principal stress, which we have denoted by σ_1. It is relatively easy to find the maximum principal stress when the stresses are planar using the transformation formulas. When all the stress components are present, finding σ_1 is more difficult. Finite element programs typically compute the principal stresses at nodal points.

According to this criterion, the material will not fail if the maximum principal stress is compressive or tensile but less than the critical level of tensile stress, adjusted as usual by say a factor of safety. However, brittle materials can also fail in compression. Such failure criteria are more complicated to implement.