<u>Informal</u> Notes on Elasticity for 24-262

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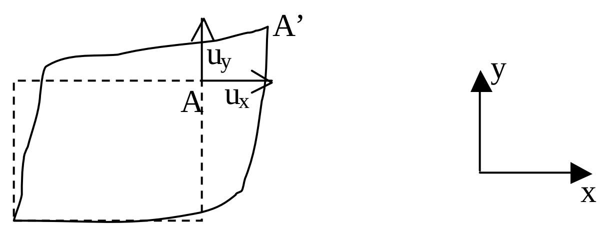
INTRODUCTION

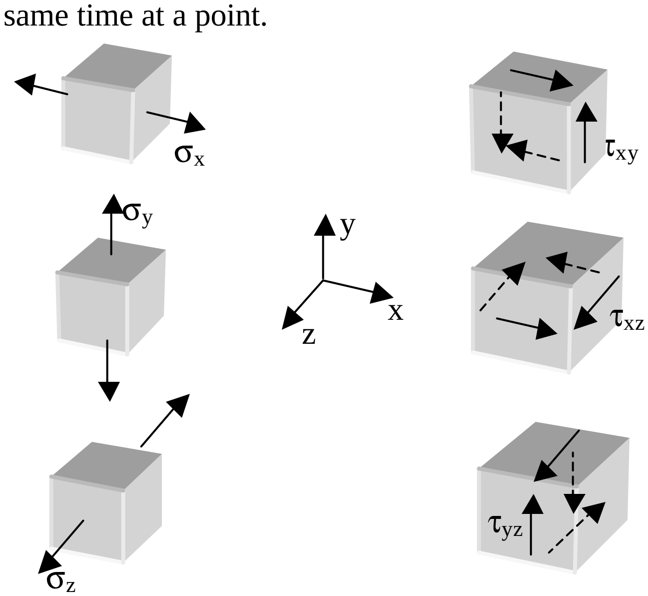
We use elasticity theory to analyze the stresses in bodies when the shape of the bodies and/or the loading does not fit into the simple cases of axial loading, torsion or bending. In using elasticity theory, we are still interested in finding stresses and displacements.

QUANTITIES: DISPLACEMENTS, STRESSES, AND STRAINS

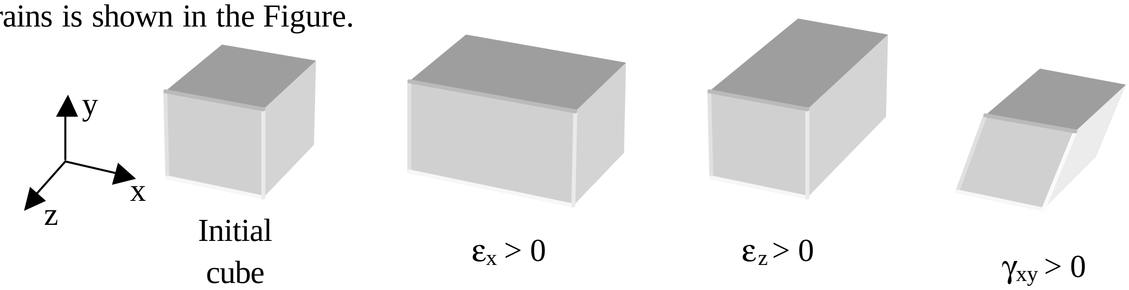
Under the action of loads which are in equilibrium, the body displaces from its initial shape. The displacements refer to the motions of a point of the body in the three coordinate directions – we call them u_x, u_y, and u_z. These motions are from the undeformed shape into the deformed shape. For example, in the diagram, the corner point A of the initially rectangular body displaces to the new position A'. (The loads are not shown.) This point has moved u_x in the x-direction and u_y in the y-direction.



The stress at each point in a body is the force per unit area acting on the 6 faces of a tiny cube of material at that point. Since this cube is in equilibrium, the net force and net moment on it are zero. Even though there are three components of force on each of the six faces, we have shown in class that equilibrium of the cube implies that there are only 6 independent stresses, shown acting individually on the cubes below. These are called components of stress and are shown below. Normal stresses (σ_x, σ_y, σ_z) act perpendicularly to a face; shear stresses (γ_{xy}, γ_{xz}, γ_{yz}) act parallel to a face. Even though each stress is drawn as a single force on each face, the force per unit area actually acts uniformly over the face. We show the different components of stress separately, but any and all can act at the same time at a point.



The deformation of such a cube at a point is described by the strain. There are 6 independent types of strain. There is stretching in each of the x-, y- and z-directions. These are normal strains, denoted by ε_x, ε_y, ε_z. Two of these strains are shown below in the Figure. There are three ways of shearing the cube with respect to the three axes: in the x-y plane (γ_{xy}), in the x-z plane (γ_{xz}), and in the y-z plane (γ_{yz}). One of these shear strains is shown in the Figure.



POSING AN ELASTICITY PROBLEM

Elasticity theory permits one to find the stresses and displacements in the body, but first one must describe the physical situation clearly in the language of elasticity. One must specify:

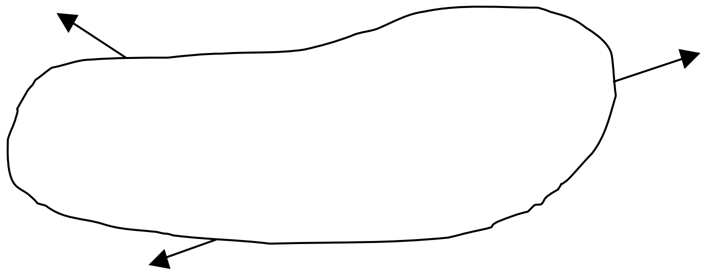
(1) the geometry (shape and size) of the body (this is the shape before any loads are applied)
(2) the material of the body (usually this means specifying the Young's modulus E and the Poisson ratio ν of the material)
(3) external influences on the body (besides gravity if it is relevant, this usually means describing what is happening on the boundary)

Below we will describe the equations that are solved in elasticity. While the equations of elasticity can sometimes be solved directly, nowadays they are more often solved using the finite element method. In this class, we are using the general-purpose commercial finite element program ANSYS.

The most challenging aspects of using elasticity to solve practical problems are: deciding what to analyze (how much of a given physical situation to analyze) and deciding what are the external influences on the part that you do decide to analyze. The same questions sometimes make statics challenging: which body should you isolate and what are all the forces in acting on the body you decided to isolate?

Elasticity differs from statics though in two important respects. First, in elasticity we are concerned about the body's deformation. Of course, that has also been a concern in axial loading, torsion or bending. Second, in elasticity when we describe the external influences on the body, we give a *very detailed description* of what goes on *at every point* of the boundary of the body. These external influences are called boundary conditions.

We consider this process of prescribing boundary conditions for one class of problems in which the body is flat and relatively thin, and the loads are in the plane of the body. This is shown below.



DESCRIBING CONDITIONS ON THE BOUNDARY

Here first is a summary: to describe completely what the surroundings are doing to this body, we must prescribe:

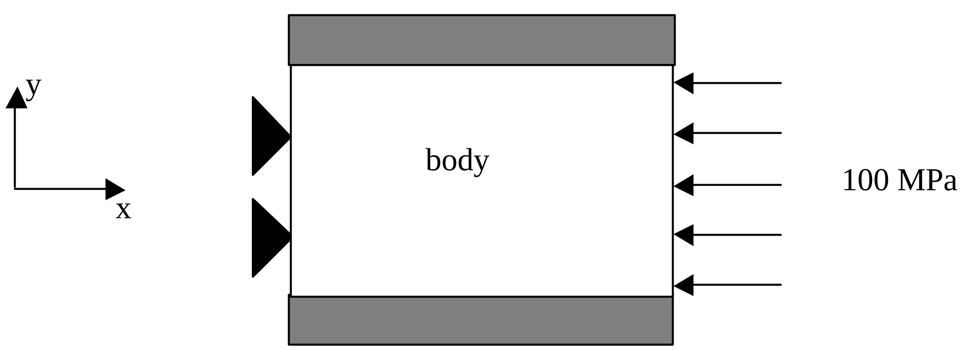
1. displacement in the x-direction **OR** force in the x-direction *at each point* along the boundary of the body

<u>and</u>

2. displacement in the y-direction **OR** force in the y-direction *at each point* along the boundary of the body

Notice: we must specify either displacement or force, but not both, in each of the x- and y-directions.

Here is an example that illustrates several possible boundary conditions. The body is a rectangular plate. At the left, the body is anchored at two points. There is uniform pressure acting on the right face. Above and below the body are rigid, immobile blocks (shaded) which keep the body from expanding, but allow the body to slide laterally.



Here is how we translate these physical conditions into boundary conditions that elasticity theory understands.

At the right: We saying nothing *directly* about the motion of this boundary; that is, we are saying nothing about displacements. We *are* saying that there is a force per unit area of magnitude 100 MPa acting in the negative x-direction. We are also tacitly saying that there is no force acting in the y-direction. To the extent that the material at this boundary would like to move up or down, we are doing nothing to stop it. So on this boundary we are prescribing force (or equivalently force per area) in both the x- and y-directions.

At the top and bottom: We saying no displacements in the y-direction are allowed by the rigid immobile blocks. That is, $u_y = 0$. However, these blocks put no restraint on the horizontal motion of points of the body which contact the block. This is saying that there is no friction between the body and blocks, or zero force in the x-direction. So on this boundary we are prescribing force in the x-direction (to be zero) and displacement in the y-direction (to be zero).

At the left: When we say the body is anchored at these two points, it means that there is no motion of the body *at those points* in the x-direction and no motion in the y-direction. Consider the rest of the left boundary. We show nothing acting on those parts of the boundary. Those parts of the boundary are intended to be free, which means there are no external forces acting on them. For those parts of the left boundary, the forces in x- and y-directions are prescribed to be zero. So on this left boundary we have two regions: at the anchors we are prescribing the displacement in the x- and y-directions (to be zero), and at the rest of the boundary we are prescribing force in the x- and y-directions (to be zero). [Note that we usually don't fix a body a one or two points; it will lead to very high stresses there.]

To sum up in a more graphic way: at each point of the boundary, one must prescribe either (i) how hard you pull or push on the point (a force) or (ii) how far you have moved the point, but you cannot prescribe both. You must prescribe one of these for each of the coordinate directions. Were you to prescribe both force *and* displacement in a given direction, then they could be inconsistent with each other, given the rest of the conditions on the boundary. Besides, it is unnecessary to prescribe both, provided you do prescribe conditions on the entire boundary.

EQUATIONS OF ELASTICITY

The ideas behind the equations of elasticity are familiar, but the equations themselves may look complicated. The same three ideas from mechanics of materials arise:

Equilibrium: the stresses vary from point to point so that they are in equilibrium with each other and with any forces applied on the boundary. This results in 3 partial differential equations involving the 6 components of stress.

When there is a planar state of stress (only σ_x, σ_y, and τ_{xy}), then there are only two such equations and they are as follows:

$$\frac{\partial s_x}{\partial x} + \frac{\partial t_{xy}}{\partial y} = 0 \qquad \frac{\partial t_{xy}}{\partial x} + \frac{\partial s_y}{\partial y} = 0$$

Material Law: the stresses and the strains are related by the elastic moduli. This results in 6 algebraic equations involving 6 components of stress and 6 components of strain. For a planar state of stress, the three components of stress and strain in the plane are related by

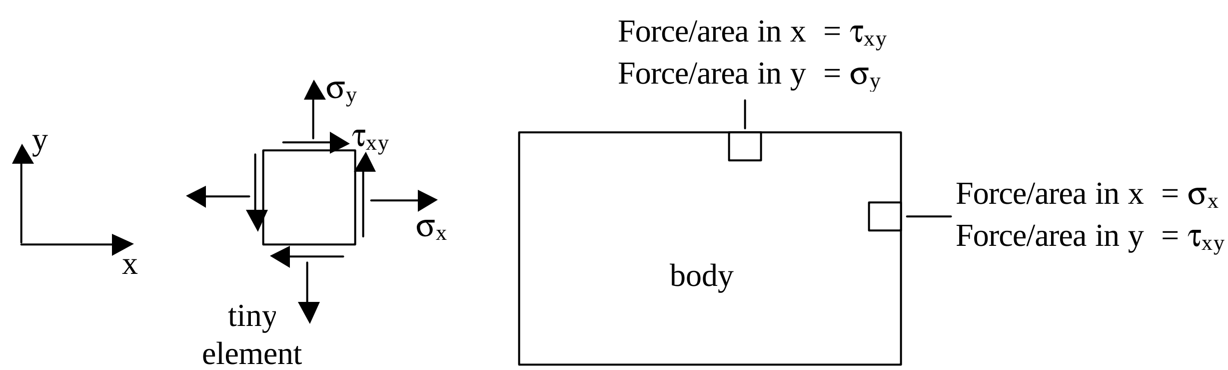
$$e_x = \frac{1}{E}\left[s_x - n s_y\right] \qquad e_y = \frac{1}{E}\left[s_y - n s_x\right] \qquad g_{xy} = \frac{1+n}{E}\left[t_{xy}\right]$$

Geometric Compatibility: the strains are related to variations in the displacements (these relations generalize the basic relation from axial loading $\varepsilon = \delta/L$ and a similar one for shear strain). This results in 6 partial differential equations involving the strains and the displacements. For a planar state of stress, there are 3 relations between strain and displacement

$$e_x = \frac{\partial u_x}{\partial x} \qquad e_y = \frac{\partial u_y}{\partial y} \qquad g_{xy} = \frac{1}{2}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right]$$

In total, there are 15 equations for the 15 stresses, strain and displacement. For a plane stress problem there are 8 equations for the three stresses (σ_x, σ_y, and τ_{xy}), three strains (ε_x, ε_y, and γ_{xy}) and 2 displacements (u_x and u_y). In addition, there are boundary conditions, which we have described above for the case of plane stress problems.

One point to add about boundary conditions is the following. When the conditions involve specifying the force on the boundary, then this can often be stated in terms of force per unit area on the boundary. Then, if the boundary is parallel to either the x- or y-axis, then force per area is the same as one of the stress components.

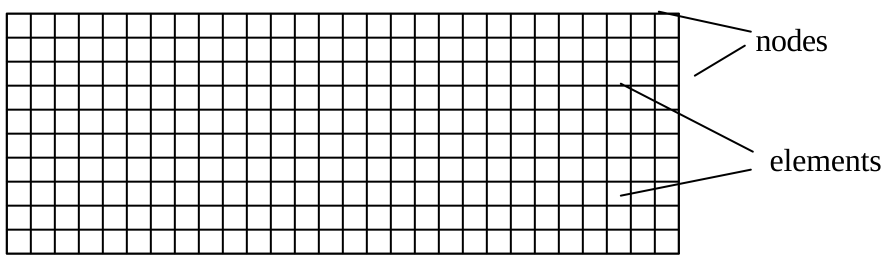


FINITE ELEMENT METHOD

In the finite element method, the computer solves the equations of elasticity. It is still necessary to describe the elasticity problem unambiguously to the program. This includes specifying the body, the material, and the boundary conditions.

In the finite element method the equations of elasticity are not solved at every point but in discrete or finite elements (rather than infinitesimal elements). The accuracy will depend on how many elements you use – the more elements, the more accurate, although eventually the improvement diminishes with increasing number of elements. It takes experience to know how many elements and how to distribute the elements in the body.

This breaking up of the body is called meshing. It results in a set of elements (usually quadrilaterals for planar problems) and a set of nodes where the element boundaries intersect (see Figure).



Besides describing the mesh, you need to specify what kinds of elements to use. For example, with some of the problems discussed here, you would specify that the elements are to be two-dimensional (plane stress) elements.

The boundary conditions are prescribed after you have a mesh. Remember that with boundary conditions, you have to prescribe the displacement or the force at each point of the boundary. With the finite element method, you need to prescribe the displacement or the force just at each node on the boundary. (Thus if there is some distribution of force on the boundary, then there must be a sufficient density of nodes on the boundary to capture the distribution.)

The finite element method offers one convenience. If you say specify nothing regarding a particular boundary node (that is, you do not prescribe a force or displacement), then the program will take there to be no external force acting on that node.