21-228 Week 4 Notes

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1 Generating Functions

1.1 Selecting Fruit

We can use these as another tool for counting functions. More generally they are an alternative method for dealing with sequences.

First, let us consider selecting objects. Suppose we have two apples, three nectarines, and for plums. We wish to give a compact description of all possible fruit selections with at least one of each fruit (disregarding order).

Let A stand for "apple", N for "nectarine" and P for "plum"

Then we might let $ANNPPP$ stand for "one apple, two nectarines, and three plums. If we do this, we see a useful shorthand – rewrite this as AN^2P^3. Also, a selection denoted ANP is the same as one denoted PNA. Thereforew e have ANP or ANP^2, but not both at the same time. So we can think of our possible fruit selections as being multiplications of the following form:

$$(A + A^2)(N + N^2 + N^3)(P + P^2 + P^3 + P^4) \tag{1.1}$$

And we can expand the resulting polynomial to get the result of making this substitution. If we want all arrangements of, say, six fruits then, we substitute x in for each of A, P, and N, and expand the resulting polynomial. The coefficient of the x^6 term is the desired answer.

In the above example, each fruit was the same in the end – since we just wanted six pieces, it didn't matter which fruits were included. Thus, each one has the same "weight" when we tally up our total. Now, suppose we want a more "weighted" counting. For instance, let's say a plum has 20 calories, a nectarine has 40 calories, and an apple 60 calories. Say we want all fruit selections with 200 calories.

Then, in a sense, a plum would have "weight" 20, a nectarine would have "weight" 40, and an apple would have "weight" 60. So, to get the total number of calories, we would instead subsitute x^{20} for P, x^{40} for N, and so forth. After expanding the polynomial, we take the coefficient of the x^{200} term.

In general, then, when we want the number of ways to produce a "weighted" sum of k in this manner, we expand the polynomial resulting by substituting an object of weight w with x^w and then find the coefficient of x^k. Thus, we can produce a sequence a_i, which is the number of ways to choose objects to get weighted sum i. From now on, we say that a selection has "value" i (denoted $v(i)$) if the weight has

Now let's make things a bit more interesting. Suppose we have infinitely many of each piece of fruit, and furthermore, we needn't take at least one of each. We then need to replace (1.1) with

$$(A^0 + A^1 + \dots)(N^0 + N^1 + \dots)(P^0 + P^1 + \dots)$$

And then the x^k selection of *this* polynomial, when expanded is the desired value. But I just lied when I said polynomial, of course – the above values have infinitely many terms. But the approach is the same. Multiply out the expression to the nth term (this can be done even though each factor has infinitely many terms – why?) to get the x^n coefficient. Replacing A, N, and P by x thus gives us what we want.

1.2 Definition of Generating Function

Suppose that c_n is a sequence. Then the generating function for c_n is $\sum_{a=1}^{n} c_n x^n$. The examples we went through above calculating the generating function, for instance, the sequence a_i where a_i is the way to select fruits to get i pieces of fruit.

However, we should view these functions as "expressions" rather than a "power series" that many of you may have seen in a calculus course. In this context, we therefore use the term "formal power series".

1.3 Product Principle for Generating Functions

Theorem 1.1 (Product Principle for Generating Function). *Let v and w be nonnegfative integer valued functions defined on sets S and T. Let a_i be the number of objects s in S with $v(s) = i$ and b_i be the number of objects t in T with $w(t) = i$. Then*

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{i=0}^{\infty} b_i x^i\right)$$

Is the generating function for the sequence c_j, where c_j is the number of ordered pairs $(s,t) \in S \times T$ with $v(s) + w(t) = j$.

Proof. A result of the sum and product principle we learned earlier. $\square$

1.4 Generating function for multisets

Theorem 1.2. *The generating function for the number of k element multisets of an n element set is $(1-x)^{-n}$.*

Proof. By the earlier techniques $\square$

2 Applications to Partitions and I/E

2.1 Change-Making

Suppose that we have a pile of nickels, dimes, and quarters, and we wish to make change for a dollar. In what ways can we do this?

Let's apply the technique of last time. We let N stand for nickel, D for dime, and Q for quarter.

Then we have

$$(N^0 \oplus N^1 \oplus \dots)(D^0 \oplus D^1 \oplus \dots)(Q^0 \oplus Q^1 \oplus \dots)$$

We want a "weighted" sum of these to reach 100. So we replace N by x^5, D by x^{10}, and Q by x^{25}.

We then get the following generating function:

$$(x^0 + x^5 + \dots)(x^0 + x^{10} + \dots)(x^0 + x^{25} + \dots)$$

Which simplifies to, by the formula for geometric sum:

$$\frac{1}{1-x^5}\frac{1}{1-x^{10}}\frac{1}{1-x^{25}}$$

There's a problem here, though – how the hell do we find the x^{100} coefficient in the corresponding power series?

Let c_i be the number of ways to make change for i cents using nickels, dimes, and quarters. To find c_{100}, we first work with simpler problems. W e ask what we can do if there are only nickels and dimes. So let b_i be the number of ways to make change for i cents using only nickels and dimes, a_i be the ways of doing so with only nickels.

Techniques similar to the above show that

$$(1-x^5)^{-1}(1-x^{10})^{-1} = \sum_{i=0}^{\infty} b_i x^i \tag{2.1}$$

Also, we know that

$$(1-x^5)^{-1} = \sum_{i=0}^{\infty} a_i x^i$$

Let us multiply both sides of (2.1) to get:

$$(1-x^5 -1) = (1-x^{10}) \sum_{i=0}^{\infty} b_i x^i$$

Substituting, we get:

$$\begin{aligned} \sum_{i=0}^{\infty} a_i x^i &= \sum_{i=0}^{\infty} b_i (x^i - x^{i+10}) \\ &= \sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} b_i x^i - \sum_{i=0}^{\infty} b_i x^{i+10} \end{aligned}$$

Now, let $b_{-10} = b_{-9} = \cdots = b_{-1} = 0$, we may rewrite as

$$\sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} b_i x^i - \sum_{i=0}^{\infty} b_{i-10} x^i$$

By convention, we set $a_0 = b_0 = c_0 = 1$ – this is more for notational convenience – and it makes a certain sense – the change consisting of nothing at all gives us 0 cents. Dividing by x_i now tells us that

$$b_i = a_i + b_{i-10}$$

So now we may easily calculate all of the b_i, since the a_i are easy to find.
How do we build ot the c_i? we perform a similar trick.
We get

$$\left(\sum_{i=0}^{\infty} b_i x^i \right) (1-x^{25})^{-1} = \sum_{i=0}^{\infty} c_i x^i$$

And as before, we isolate the $b_i x^i$ to get:

$$\sum_{i=0}^{\infty} b_i x^i = (1 - x^{25}) \sum_{i=0}^{\infty} c_i x^i$$

Again, we let $c_i = 0$ when $i < 0$, so

$$\sum_{i=0} b_i x^i = \sum_{i=0}^{\infty} (c_i - c_{i-25}) x^i$$

Thus, $c_i = b_i + c_{i-25}$.

2.2 Generating Functions for Integer Partitions

So, we have essenetially found a way for generating all partitions of a number into parts of size 5, 10, and 25. We can apply techniques analagous to those above to prove the following statement:

Theorem 2.1. *Let a_i be the number of ways to partition the integer i using integers $1, 2, \ldots, n$. Then the generating function for a_i is*

$$\prod_{j=1}^{n} \frac{1}{1 - x^j}$$

If we want the total number of all partitions, we take the above over all integers – that is:

$$\prod_{j=1}^{\infty} \frac{1}{1 - x^j}$$

3 Application to Binomial Coefficients

Suppose that we have n different candy bars and want to select k of them. We know from previous methods that the tnumber of ways to do this is $\binom{n}{k}$. We may also get the answer using generating function. Each candy bar may get chosen 0 or 1 times. Let a_k be the number of ways to choose k pieces of candy out of n. Then the generating function for a_k is $(1 + x)^n$. (Why?). Thus, another proof that $(1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$.

Now, let's extend the definition of binomial coefficients:

Remember we noted before that $\binom{n}{k} = \frac{(n)_k}{k!}$. Let m be a positive number. Then

$$
\begin{aligned}
\binom{-m}{k} &= \frac{\prod_{i=0}^{k-1}(-m-i)}{k!} \\
&= (-1)^k \frac{\prod_{i=0}^{k-1} m+i}{k!} \\
&= (-1)^k \binom{m+k-1}{k}
\end{aligned}
$$

So we have just proved:

Theorem 3.1. *If $m, k > 0$, then:*

$$\binom{-m}{k} = (-1)^k \binom{m+k-1}{k}$$

We can apply the binomial theorem to get:

$$(1+x)^{-m} = \sum_{k=0}^{\infty} \binom{-m}{k} x^k$$

3.1 Apps to Inclusion/Exclusion

We may solve this using inclusion/exclusion (how?).

Let's say we wish to give 10 pieces of candy to three children so no child gets more than four pieces.

As seen above, the generating function is $(1+x+x^2+x^3+x^4)^3$.

We wish to find the x^{10} coefficient to this. We can find this by noticing

$$(1-x)(1+x+x^2+x^3+x^4) = 1-x^5$$

So

$$(1+x+x^2+x^3+x^4) = \frac{1-x^5}{1-x}$$

The generating function is the cube of the above, so cubing both sides yields

$$
\begin{aligned}
(1+x+x^2+x^3+x^4)^3 &= \frac{(1-x^5)^3}{(1-x)^3} \\
&= (1-x^5)^3(1-x)^{-3} \\
&= (1-3x^5+3x^{10}-x^{15}) \sum_{i=0}^{\infty} \binom{3+i-1}{i} x^i \\
&= (1-3x^5+3x^{10}-x^{15}) \sum_{i=0}^{\infty} \binom{2+i}{i} x^i
\end{aligned}
$$

The term involving x^{10} is

$$\binom{2+10}{10}x^{10} - 3x^5\binom{2+5}{5}x^5 + 3x^{10}\binom{2+0}{0}x^0$$

The coefficient works out to $\binom{12}{10} - 3\binom{2+5}{5} + 3\binom{2+0}{0}$ – which is $66 - 3(21) + 3(1) = 3$.

4 Recurrence Relations and Generating Functions

4.1 The Idea of a Recurrence Relation

One of the reasons why generating functions are an important tool is that we can use them to solve recurrences.

For instance, we may have $a_n = 2a_{n-1}$, or $a_n = 2a_{n-1} - a_{n-2}$, or so forth. If each term is a multiple of some a_i, we say the recurrence is homogeneous. We say a recurrence is of *order* k if the recurrence uses the terms a_n through a_{n-k} (i.e. the range of elements used spans at most $(k+1)$ elements).

4.2 How Generating functions are Relevant

Example: Find the generating function for a_i, where a_i is the number of elements of an i element set.

We know that $a_0 = 1$, and $a_{i+1} = 2a_i$ whenever $i \geq 0$.

We multiply both sides by x_i and sum over all i to get:

$$\sum_{i=0}^{\infty} a_{i+1}x^{i+1} = \sum_{i=0}^{\infty} 2a_i x^{i+1}$$

So:

$$\sum_{i=0}^{\infty} a_{i+1}x^{i+1} = 2x\sum_{i=0}^{\infty} a_i x^i$$

Now, consider the left hand side of the above equation. Since $a_0 = 1$, we have:

$$\sum_{i=0}^{\infty} a_i x^i - a_0 = 2x\sum_{i=0}^{\infty} a_i x^i$$

So we have

$$a_0 = \sum_{i=0}^{\infty} a_i x^i - 2x \sum_{i=0}^{\infty} a_i x^i$$

So,

$$(1-2x) \sum_{i=0}^{\infty} a_i x^i = a_0$$

So, we have

$$\frac{a_0}{1-2x} = \sum_{i=0}^{\infty} a_i x^i = a_0 \sum_{i=0}^{\infty} (2x)^i$$

So, it follows that $a_i = a_0 2^i$.

Therefore, since $a_0 = 1$, we have $a_i = 2^i$.

Now, we could easily have done this problem without generating functions, so let's now do something a bit different. What if we have $a_0 = 0$, but now $a_{n+1} = 2a_n + (n+1)$?

Then we can do the same technique:

$$\sum_{i=0}^{\infty} a_{i+1} x^{i+1} = \sum_{i=0}^{\infty} 2a_i x^{i+1} + (i+1)x^{i+1}$$

So,

$$\sum_{i=0}^{\infty} a_{i+1} x^{i+1} = 2x \left(\sum_{i=0}^{\infty} a_i x^i + (i+1)x^i \right)$$

Now, since $a_0 = 0$, the above is the same as:

$$\sum_{i=0}^{\infty} a_i x^i = 2x \left(\sum_{i=0}^{\infty} a_i x^i + (i+1)x^i \right)$$

Moving the first term onto the lefthand side gives us

$$(1-2x) \sum_{i=0}^{\infty} a_i x^i = 2x(\sum_{i=0}^{\infty} (i+1)x^i)$$

The sum on the right hand side can be found by integrating – upon integrating we get $\sum_{i=1}^{\infty} x^i$, which is $\frac{x}{1-x}$. Taking the derivative again gives us $\frac{1}{(1-x)^2}$.

So we have

$$(1-2x)\sum_{i=0}^{\infty} = \frac{2x}{(1-x)^2}$$