21-228 Week 4 Notes

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1 Partitions and Stirling Numbers

Let's try and solve more counting problems. Let's say now we have m objects, and we wish to partition them into n classes. This is different from broken permutations in that now we do not care about the order of objects within a given class. We do not know how to get an exact formula for this number, we can make some statements about this value.

Definition 1.1 (Stirling Numbers/Second Kind). *We denote by $S(m, n)$ the number of partitions of an m-element set into n classes. These are called Stirling numbers of the second kind.*

Theorem 1.2. $S(m, n) = S(m-1, n-1) + nS(m-1, n)$

Proof. Let us consider an M element set, and let $m \in M$ be fixed. Either m is in a partition by itself or m is in a partition with some other elements – namely, the remaining $n-1$ elements. If m is by itself then the remaining $n-1$ elements are partitioned into $n-1$ classes – there are $S(m-1, n-1)$ ways of doing this. If m is grouped with something else, the remainder of the $m-1$ elements are also split into n classes, and m may be in any of the n classes. Thus there are $nS(m-1, n)$ possibilities. By the sum principle, we get $S(m, n) = S(m-1, n-1) + nS(m-1, n)$. □

There is also another recursive relationship:

Theorem 1.3.

$$S(m, n) = \sum_{j=0}^{m-1} \binom{m-1}{j} S(j, n-1)$$

Proof. Consider a partition of an m-element set into n classes. Now we remove the class containing m – this may contain anywhere from 1 to m elements. Then the remaining elements form a set of j elements, where j is anywhere from 0 to $m-1$ in size. There are, for any given j exactly $binomm - 1j$ ways of picking the j elements that remain, and these can be partitioned in $S(j, n-1)$ ways. Each of these partitions forms its own unique parititon of an m-element set into n classes. Thus we get, from the sum principle $\sum_{j=0}^{m-1} \binom{m-1}{j} S(j, n-1)$. $\square$

1.1 Onto Functions/Stirling Numbers

Let us suppose that we have a partition $Q = \{C_1, \ldots, C_n\}$ into n classes, and a one-to-one function G from Q to [], we can define a function f from $[m]$ onto $[n]$ by $f(x) = i$ iff $x \in C_i$. So, the number $F(m, n)$ of functions from an m element set onto an n element set is $S(m, n)n!$, since we may rearrange the classes in any way to get a different function.

2 Stirling Numbers of the First Kind

Stirling numbers were intended to describe polynomials instead of partitions. Let's look at Stirling Numbers of the First Kind.

Consider the number of one-to-one functions from an m-element set to an n-element set – this number is $\frac{n!}{(n-m)!} = \prod_{i=1} m(n-i+1)$. Thus, if we consider n to be the variable, we have a polynomial in n of degree m. So, we can rewrite the polynomial as follows:

$$\frac{n!}{(n-m)!} = \sum_{j=0}^{m} s(m, j)n^j$$

These $s(m, j)$ are the *Stirling numbers of the first kind.*

Note that above, we have rewritten $\frac{n!}{(n-m)!}$ into $\prod_{i=1}^{m}(n-i+1)$. This second form is a perfectly well-defined expression even when n is not an integer. Thus, it makes sense to also talk about the value. A notation that the book has been using, which I have been avoiding, is using $(n)_m$ to mean the number of one-to-one functions from an m element set into an n element set. We now have a well-defined way of also describing what $(x)_m$ is, for when x is an arbitrary real number instead of just an integer. Note also that the coefficients of the resulting polynomial are unchanged.

Definition 2.1 (Factorial Power). *We call $(x)_m$ a factorial power of x of degree m, and it is $\prod_{i=1}^{m}(n-i+1)$.*

Theorem 2.2. *For any real number x, $(x)_m = \sum_{j=0}^{m} s(m, j)x^j$.*

Proof. A polynomial in x of degree m is completely determined by its value at any $m+1$ distinct values of x. Just take any $m+1$ integers n that are larger than m, and these are the distinct values of x needed. Another proof arises from the fact that whether x is an integer or real has no effect on the coefficients we get after manually multiplying out the product. $\square$

2.1 Stirling's Triangle of the First Kind

Just as the $S(m,n)$ had a noice recursive relation, so do the $s(m,n)$. In particular, we have:

Theorem 2.3. $s(m,n) = s(m-1,n-1) - (m-1)s(m-1,n)$

Proof.

$$\sum_{j=0}^{m} s(m,j)x^j = (x_m)$$

$$\begin{aligned}
&= (x)_{m-1}(x-m+1) \\
&= \sum_{i=0}^{m-1} s(m-1,i)x^i(x-m+1) \\
&= \sum_{i=0}^{m-1} \left(s(m-1,i)x^{i+1} - (m-1)s(m-1,i)x^i\right)
\end{aligned}$$

Then two terms on this last value contain x^j - namely $s(m-1,j-1)x^j$ and $-(m-1)s(m-1,j)x^j$. $\square$

3 Stirling Numbers of the Second Kind as Polynomials

So we have looked at Stirling's numbers of the first kind as polynomials. It turns out that there is an interesting polynomial for which Stirling's numbers of the second kind form coefficients. In particular we have the following theorem:

Theorem 3.1.

$$n^m = \sum_{j=0}^{n} S(m,j)(n)_j$$

Proof. We have seen that there are $S(m,j)j!$ surjections from an m element set onto a j element set. Also, each function from an m element set to an n element is a surjection onto some subset J of the n element set. And if j is fixed, there are $\binom{n}{j}$ possible ranges of size j. Therefore, we get:

$$n^m = \sum_{j=0}^{n}\binom{n}{j}S(m,j)j!$$

$$= \sum_{j=0}^{n}\frac{n!}{(n-j)!}S(m,j)$$

& $\sum_{j=0}^{n} S(m,j)(n)_j$

The first line in the above computation is the most interesting. n^m is the total number of functions from an m element set onto an n element set. Each term of the sum on the RHS represents the number of functions that are onto some j element subset of the n-element set. □

Now, replacing n by x above is not so easy because then we would have to sum up to x, which is no longer guaranteed to be an integer. But we can work around this by noticing that only terms above where $j < m$ are relevant, and also only terms where $j < n$ are relevant, so that we may instead sum from $j = 0$ to m instead of $j = 0$ to n. Then replacing n with x is simple, and we have:

Theorem 3.2. $x^m = \sum_{j=0}^{m} S(m,j)(x)_j$

4 Bell Numbers – the total number of Partitions of a Set

Let B_m stand for the total number of partitions of an m-element set. Then, clearly, $B_m = \sum_{n=0}^{m} S(m,n)$.

Furthermore, we have the following theorem:

Theorem 4.1. *For $m > 0$, $B_m = \sum_{j=0}^{m-1} \binom{m-1}{j} B_j$*

Proof. See Bogart. The proof given is a lot of algebra. As a HW, I will ask you to try and find a more intuitive proof. □

5 Partitions of Integers

Now, we consider distributing identical objects to identical recipients. We consider now the concept of a multiset. A multiset is a set, except now we may allow an element to appear more than once. Formally, we have:

Definition 5.1 (Multiset). *A multiset M is a set S together with a function $f : S \to \mathbb{N}$. We say that $f(s)$ is the <u>multiplicity</u> of s in M*

So, the set S consists of the elements used, and the function f tells us how many times s appears in the multiset.

We want to ask, given a number m, how many ways can we add up other natural numbers and get n. We do not care about the order in which these numbers occur. So this is similar to placing identical pieces of candy into identical bags – or placing identical books into identical shelves.

So, if we are given a number m and numbers $a_1, \ldots, a_n$ so that $a_1 + \cdots + a_n$, we want to know how many ways the a_i can be chosen so that they are decreasing (that is, $a_{i+1} \leq a_i$ for each i), and so that the a_i's sum to m.

5.1 Number of Partitions of m into n parts

Bythe number of parts of a parition, we mean the size of the multiset used to describe it. So we may either sum the multiplicities, or measure the length of the decreasing list. Let us use $P(m, n)$ to describe the number of partitions of m into n numbers. Then we have:

Theorem 5.2.

$$\sum_{i=1}^{n} P(m, i) = P(m + n, n)$$

Proof. First consider a partition of $m + n$ into n parts. This can be turned into a partition of m into at most n parts by deleting 1 from each of the n parts of the partition of $m + n$, and removing the parts that are now 0.

Now, consider a partition of m into $i \leq n$ parts. We can turn this into a partiton of $m + n$ into n parts by adding 1 to each of the i parts, then adding $n - i$ parts of 1. The two constructions given reverse each other – so we have formed an invertible map from the set of partitions of m into at most n parts into the set of partitions of $m + n$ into exactly n parts. □

Corollary 5.3. *For $n < k$, $P(k, n) = \sum_{i=1}^{n} P(k - n, i)$.*

6 Principle of Inclusion and Exclusion

6.1 Size of Union of Two Sets

If we have two sets A and B, we get:

Fact 6.1. $|A \cup B| = |A| + |B| - |A \cap B|$

Proof. If we add the sizes of A and B, we count the elements in $A \cap B$ twice – therefore we subtract $|A \cap B|$ as an adjustment. □

6.2 Size of Union of Three Overlapping Sets

Suppose that we have three sets $|A|$, $|B|$ and $|C|$. We would like to compute the union.

Now, if we simply take $|A| + |B| + |C|$, we are double counting the elements in $A \cap B$, $B \cap C$, and $A \cap C$ twice, and further, those of these that are in $A \cap B \cap C$

are counted three times. So we subtract $|A \cap B|$, $|B \cap C|$, and $|A \cap C|$, but now we have "decounted" those elements in $A \cap B \cap C$ three times when we only wanted to do so twice – so now we add $|A \cap B \cap C|$ to get:

Fact 6.2.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

6.3 The number of Onto Functions

We still do not have an exact formula for the number of surjections from an m element set onto an n element set. If the n is small, however, the computation can be made directly, using the Inclusion/Exclusion principle. We count the number of total functions, then for each i, subtract the number of functions that skip i. But then we count those that skip two different numbers twice, so we add back in those that skip two numbers. Then we subtract those that skip three numbers, and so forth...

7 Inclusion/Exclusion Principle and Properties

Let S be a set of properties and A be a set of objects. If $T \subseteq S$ let $N_{\geq}(T)$ be the set of all objects in A that contain properties in T (together with possibly other properties). Let $N_{=}(T)$ be the set of all objects in A that contain *exactly* those properties in T.

Then we have the following:

Theorem 7.1 (Inclusion/Exclusion Principle). *In the above notation, we have:*

$$|N_{=}(T)| = \sum_{J=T}^{S} (-1)^{|J|-|T|} N_{\geq}(J)$$

Questions (work these out yourselves for practice):

1. Charles, Mary, Alice, and Pat are sitting in the four chairs at a library table. They go to class and return to the same table. In how many ways may they sit so that Pat is in the same chair she had before? In how many ways may they sit so that *only* Pat is in the same chair as before?

2. How many onto functions are there from $[m]$ onto $[n]$.

3. A used car dealer has 18 cars on the lot. 9 of them have automatics, 12 have power steering, and 8 have power brakes. 7 have both auto trans and power steer, four have auto trans and power brakes, and 5 have power steering and power brakes. Three have all three properties. How mnay cars have *only* automatic transmission? How many cars are "stripped"?

4. A group of couple sits around a circular table for a group discussion of marital problems. How many ways may we seat the couples so that no husband and wife sit together? Note that two arrangements are the same if we can rotate the table from one arrangement and get the other arrangement.