1 Introduction

Graph theory is one of the most in-demand (i.e. profitable) and heavily-studied areas of applied mathematics and theoretical computer science. May graph theory questions are applied in this way – others are more of a "pure mathematical" flavor.

One of the most powerful aspects of graph theory is that graphs can be used to model several real-world situations:

- How can we lay cable at minimum cost to make every telephone reachable from every other?
- What is the fastest route between two given cities?
- How can we assign n jobs to n people to get the best combination of people assigned to the jobs.
- How many layers does a computer chip need so that wires in the same layer don't cross?
- How can the season of a sports league be scheduled into the minimum number of weeks?
- In what order should a traveling salesman visit cities to minimize travel time?
- Can we color the regions of every map using four colors so that neighboring regions receive different colors?

2 What is a graph?

Definition 2.1 (graph). *A graph is an ordered pair (V, E), where V is called the vertex set and E is called the edge set. With each $e \in E$, we associate two vertices (perhaps the same) u and v, which we call the ends of e.*

Definition 2.2 (Basic Terminology). *Let $G = (V, E)$ be a graph.*

1. *If V and E are understood, then we often just denote G to be the graph. V always means a vertex set, E always means an edge set. If there is more than one graph in context, we say $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G.*
2. *Generally, u, v, w stand for vertices, e, f stand for edges.*
3. *If $e \in E$, with ends u and v, we say u and v are adjacent. We also say $e = uv$.*
4. *If the ends of e are the same, we call e a loop.*
5. *If two edges have the same ends, they are called parallel edges.*

6. *A simple graph is a graph with no loops or parallel edges. Most applications can be solved using only simple graphs, so most of our attention will be focussed on simple graphs.*

7. *A graph $G(V, E)$ is finite if V and E are finite. In this class, we will talk only about finite graphs.*

8. *In drawing a graph, we have a dot for each vertex, and a line from u to v for each edge $e = uv$*

Example: Let $G(V, E)$ be the following graph. V is the set of all cities in the graph. For each road going between two cities u, v, we have an edge e whose ends are u and v – that is, there is an edge from one city to another if there is a road between the two.

In the above definition, if u, v are the ends of e, it does not matter if u is the first end, or v is the first end. In this sense, the definition we gave above is an *undirected* graph. If we can go from u to v, then we can go from v to u. However, if we are talking about intersections in a given city, we might be able to go from u to v, but not from v to u – a road might be one-way. Or we may have a diode in a circuit. This motivates the concept of a *directed* graph:

Definition 2.3 (Directed Graph). *A directed graph, called digraph for short, is a ordered pair (V, E) – as usual V are vertices and E edges. But now, we have a function $f : E \to V \times V$. If $f(e) = (u, v)$, we call u the tail of e and v the head if e.*

So, if $f(e) = (u, v)$ we can represent this as being able to travel from u to v along edge e, but we cannot use edge e to travel from v to u. The edges now have a *direction* associated with them. If I use the term "graph", I mean an undirected graph.

3 Examples

3.1 Undirected graphs

3.1.1 The Bridges of Konigsberg

A famous problem solved by Euler is the following: we are in the city of Kongisberg, with four land masses. We have bridges between the land masses. Can we, purely by walking, cross each bridge *exactly* once? We can model the problem as a graph. We have a vertex between each land mass, and an edge for each bridge. The ends of an edge are the same as the ends of each bridge.

3.1.2 Ramsey Theory

We can model Ramsey theory using graphs. We have a vertex for each person, and there is an edge $e = uv$ for two people u and v iff the two people are acquainted.

In the case of 6 people, we know that there are always three mutual acquaintances or three mutual strangers. In terms of graph theory, this means that we can find three vertices so that for every pair within these three vertices, there is an edge between the two, or that for every pair, there is no pair between the two. This motivates a few definitions:

Definition 3.1.

A complete graph on n vertices is a simple graph $G = (V, E)$ where V is an n-element set, and for any two pairs of vertices u, v, there is an $e \in E$ whose ends are u and v. We will see later that in a given sense, there is only one such graph on n vertices, and we denote this by K_n.

If $G = (V, E)$ is a graph, then $H = (U, F)$ is a subgraph of G iff $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and each edge in H has the same ends as it does in G.

If $G = (V, E)$, and $U \subseteq E$, the graph $H = (U, F)$, where F is the set of all edges in G with both ends in U is called the subgraph of G induced by U, and is called $G[U]$.

If $G = (V, E)$ is a graph, then a clique in G is an induced subgraph $G[U]$ that is a complete graph on $|U|$ vertices. That is, each pair of vertices in U are adjacent.

An independent set U of vertices in $G = (V, E)$ is a set $U \subseteq V$ of vertices in G so that no pair of vertices in U are adjacent.

Using the above notation, we then have the following theorem:

Theorem 3.2 (Ramsey's Theorem). *For any natural numbers n, m, there is a number $R(n, m)$ so that, for any graph G with $R(m, n)$ vertices, there is a clique in G with m vertices or an independent set of n vertices in G.*

There is an alternate characterization of Ramsey's Theorem. We may *color* edges or vertices of a graph. Later I will show an example of coloring vertices, but here we will color edges.

Let us suppose we have a complete graph on $R(m, n)$ vertices. In other words, if this graph is $G = (V, E)$, we partition E into two sets $R \cup B - R$ is the set of *red* edges, and B is the set of *blue* edges. Each edge between two people that know each other is colored red. Each edge between two strangers is colored blue.

Then we can state Ramsey's Theorem as follows:

Theorem 3.3 (Ramsey's Theorem again). *For any natural numbers n, m, There is a number $R(m, n)$ so that if we color the edges of the complete graph on $R(m, n)$ vertices with two colors red and blue,then there is either an m-clique whose edges are all red, or an n-clique whose edges are all blue.*

3.1.3 Job Assignments and Bipartite Graphs

Suppose we have a set of m jobs and n people, and each person can do some of the jobs. However, we can only assign one job to any given person. Also, suppose that it is wasteful to assign more than one person to a given job, so we require that no more than one person is assigned to a job. Can we assign all of the jobs?

We can model this problem as follows. Let $G = (V, E)$ be a graph, where $V = A \cup B$ (A is the set of jobs, B is the set of people). For each $u \in A$, $v \in B$, we have an edge e with ends u, v if person v can do job u.

The problem is then reduced to seeing if we can pick a set of edges $F \subseteq E$ so for each $u \in A$, there is exactly one edge with end u and exactly one edge (not necessarily the same) with end V. Any "satisfactory" solution to this problem is called a *matching* – we are matching jobs with people.

Definition 3.4 (Matching). *1. If $G = (V, E)$ is a graph, a set $M \subseteq E$ of edges is called a* matching *if M contains no loops and no vertex $v \in V$ is an end of two edges in M.*

2. A matching saturates *$U \subseteq V$ if each $u \in U$ is an end of some edge $e \in M$.*

3. A perfect matching *in $G = (V, E)$ is a matching M in G that saturates V.*

Therefore, going back to the problem of jobs and people, we seek to find a matching that saturates A.

Note that each edge in the graph goes from A to B. Therefore, we have partitioned the vertex set into two sets so that no edge has both ends in the same set. This is a common graph:

Definition 3.5 (Bipartite Graph). *A graph $G = (V, E)$ is* bipartite *if V can be partitioned into two sets A and B so that no $e \in E$ has both ends in A or both ends in B.*

Finding a perfect matching in a graph (if it has one) is an interesting problem. It is an easier problem if we know the graph is bipartite.

4 Euler Tours

One of the oldest problems in graph theory is the Bridges of Konigsberg problem. The problem is as follows: suppose that there are several land masses, and bridges connecting these land masses. The question is, can we cross each bridge exactly once, and end up on the same land mass that we started?

We can represent this problem as a graph. Each land mass is a vertex, and each bridge is an edge. The ends of each edge are the same as the ends of the bridge that the edge represents.

Since we're talking about crossing each bridge, we can think about taking a *walk* –

Definition 4.1 (Walks). *Let $G = (V, E)$ be a graph.*

1. A walk *in G is a list $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$, where each v_i is a vertex, each e_i is an edge, so that the ends of e_i are v_{i-1} and v_i.*

2. *A trail is a walk where the e_i are all distinct.*

3. *A closed walk is a walk where $v_0 = v_k$ – that is, the walk ends at the same place that it starts.*

4. *An Euler Trail is a trail that uses every edge in G exactly once.*

5. *An Euler Tour is an Euler Trail that also is a closed walk – that is, it is a walk that uses each $e \in E$ exactly once and ends up at the same place that it starts.*

In this case, we are only interested in situations where everything is "connected" – that is, if there are two landmasses that are flat out unreachable altoegther from one another, then obviously we cannot cross all the bridges (assuming that there are bridges from each landmass).

Definition 4.2 (Connectivity). *1. Let $G = (V, E)$ be a graph, and let $u, v \in V$. We say that u and v are connected if there is a walk in G starting at u and ending at v.*

2. *G is connected if for every $u, v \in V$, u and v are connected.*

Euler discovered that in the particular case of the Bridges of Konigsberg, (see diagram in Bogart), it is impossible to cross each bridge exactly once (unless you fly or swim ...). This is because some of the land masses have an odd number of incident bridges.

Definition 4.3. *The degree of a vertex v in a graph G is the number of ends of edges in G that equal v. In the case where G has no loops, this is equal to the number of edges that are incident with v.*

Before we state the theorem, here is a lemma that will be of use:

Lemma 4.4. *Let G be a graph where every vertex in G has even degree. Then all maximum-length trails in G are closed.*

Proof. Suppose that T is a trail in G that is not closed. Then an odd number of edges incident with the last vertex v_k of T must have been used. Therefore, we may pick a new edge leaving T. □

Now, the theorem:

Theorem 4.5. *Let G be a connected graph. Then G has an Euler Tour iff every vertex in G has even degree.*

Proof. ($\Rightarrow$): Suppose that G has an Euler tour $v = v_0, e_1, v_1, e_2, v_2, \ldots, v_k = v$. Let $u \in V$ be given. Suppose that $v_j = u$. Then the edges e_j and e_{j+1} have an end at u (if $j = k$, then use e_1 instead of e_j. Therefore, it follows that

$$\deg(u) = \sum_{j \in [k] : v_j = u} 2 v_j$$

Thus, $\deg(u)$ is the sum of even numbers, and therefore must itself be even.

($\Leftarrow$): Suppose now that G is a connected graph where every vertex has even degree. Let T be a maximum trail in G. By 4.4, T must be closed. Suppose, for the sake of contradiction, that T is not Eulerian. Then there is some edge leaving a vertex v used in T that is unused. Remove the edges in T from G to form a new graph G^*. Then G^* still is a graph where every vertex has even degree (because we removed an odd number of ends from each vertex in T), and there is a trail of length at least one in the remaining graph. A maximum-length trail must thus, by 4.4, be closed. Picking a maximum length trail in G^*, and "splicing" it into the first occurrence of v in T, results in a longer trail in G. But we said that T was a maximum-length walk, which is a contradiction. $\square$