In all cases, $G = (V, E)$ is a graph. Unless otherwise stated, consider only simple graphs (graphs without loops or parallel edges).

1. We defined $u, v \in V$ to be <u>connected</u> if there is a u, v-walk in G. Show that connectedness in G is an equivalence relation on V. We call the subgraphs induced by the equivalence classes of this relation *components*.

By definition, $v \in V$ is always connected to v. If there is a u, v-walk in G, call it $(u = v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k = v)$, we may reverse the walk, as the ends of an edge are still the two neighboring vertices on the walk. Similarly, if there is a u, v-walk in G, and a v, w-walk in G, then concatenating the two walks gives a walk from u to w.

2. Prove that in a graph, the number of vertices of odd degree must be even. Hint: What is the sum of all degrees of all vertices in any graph?

First, we have, since each end has two ends, and the degree of a vertex is the number of "ends" represented by that vertex:

$$\sum_{v \in V} \delta(v) = 2\,|E|$$

Thus the sum of all degrees of all vertices of a graph is even. Now, if we sum over only the vertices of odd degree, we must still get an even number, since we would be subtracting out terms each of which are even. This can only happen if there are an even number of such vertices.

3. If a graph is simply a cycle on n vertices, how many edges does it have?

Removing an edge from such a graph is a spanning tree, with $n - 1$ edges, so the original graph had to have n vertices.

4. What is the maximum number of edges a simple graph on n vertices can have? Recall that a simple graph has no loops or parallel edges (edges with the same ends). This problem is related to the *complete graph* on n vertices, which is the graph where every pair of vertices has an edge.

There can be at most one edge for each pair of vertices, which is $n(n-1)/2$.

5. A graph is *Hamiltonian* if it has a cycle which contains all the vertices of the graph. There is no known efficient algorithm to determine if a graph is Hamiltonian, and this concept is closely related to the Travelling Salesman Problem: find the shortest cycle visiting all vertices in a graph (assuming that each edge also has a length). Find a graph with 8 vertices and 13 edges that is not Hamiltonian. Hint: Consider a complete graph on 4 vertices. This graph has 4 vertices and 6 edges.

One solution is to take two copies of K_4, and link them with an edge.

6. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. We say that G_1 and G_2 are *isomorphic* if there are bijectons $f : V_1 \to V_2$ and $g : E_1 \to E_2$ so that if the ends of $e \in E_1$ are u and v, then the ende of $g(e)$ are $f(u)$ and $f(v)$. Prove that isomorphism is an equivalence relation on the set of all graphs. Technically, there is no such thing as "the set of all graphs", but don't worry about that – just prove that the relation is reflexive, symmetric, and transitive. Isomorphisms are useful in the sense that if two graphs are isomorphic, the one graph is simply the other with the vertices and edges "renamed".

Reflexivity is accomplished by letting f, g be the identity functions.

For symmetry, suppose we have a bijection $f : V_1 \to V_2$ and $g : E_1 \to E_2$ satisfying required properties. Then f^{-1} and g^{-1} is an isomorphism from G_2 to G_1, (verify this!).

For transitivity, if $f_1 : V_1 \to V_2$, $g_1 : E_1 \to E_2$ forms an isomorphism from

G_1 to G_2, and $f_2 : V_2 \to V_3$ and $g_2 : V_2 \to V_3$ is an isomorphism from G_2 to G_3, then one can verify that $f_2 \circ f_1$ is an isomorphism from G_1 to G_3.