In all cases, $G = (V, E)$ is a graph. Unless otherwise stated, consider only simple graphs (graphs without loops or parallel edges).

1. We defined $u, v \in V$ to be <u>connected</u> if there is a u, v-walk in G. Show that connectedness in G is an equivalence relation on V. We call the subgraphs induced by the equivalence classes of this relation *components*.

2. Prove that in a graph, the number of vertices of odd degree must be even. Hint: What is the sum of all degrees of all vertices in any graph?

3. If a graph is simply a cycle on n vertices, how many edges does it have?

4. What is the maximum number of edges a simple graph on n vertices can have? Recall that a simple graph has no loops or parallel edges (edges with the same ends). This problem is related to the *complete graph* on n vertices, which is the graph where every pair of vertices has an edge.

5. A graph is *Hamiltonian* if it has a cycle which contains all the vertices of the graph. There is no known efficient algorithm to determine if a graph is Hamiltonian, and this concept is closely related to the Travelling Salesman Problem: find the shortest cycle visiting all vertices in a graph (assuming that each edge also has a length). Find a graph with 8 vertices and 13 edges that is **not** Hamiltonian. Hint: Consider a complete graph on 4 vertices. This graph has 4 vertices and 6 edges.

6. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. We say that G_1 and G_2 are *isomorphic* if there are bijectons $f : V_1 \to V_2$ and $g : E_1 \to E_2$ so that if the ends of $e \in E_1$ are u and v, then the ende of $g(e)$ are $f(u)$ and $f(v)$. Prove that isomorphism is an equivalence relation on the set of all graphs. Technically, there is no such thing as "the set of all graphs", but don't worry about that – just prove that the relation is reflexive, symmetric, and transitive. Isomorphisms are useful in the sense that if two graphs are isomorphic, the one graph is simply the other with the vertices and edges "renamed".