21-228 Homework 6 Solutions

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1. Suppose a gambler enters a casino with 1 dollar, and plays a fair game. Each time he plays the game, he wins with probability 1/2 and loses with probability 1/2. Each time he plays, $1 is at stake. Suppose that the gambler picks a goal of $$n$ that he would like to leave with. Find (and prove) a formula for the probability that he will successfully reach his goal of $$n$ before going broke. Hint: Let $f(k,n)$ be his probability of leaving with n dollars if he enters with k dollars. Your objective is to find $f(1,n)$ – what is $f(k_0,n)$ in terms of other $f(k,n)$?

Let us instead solve the generalized problem of $f(k,n)$, the probability that a gambler leaves with n dollars if he starts with 1 dollar. Then $f(0,n) = 0$ and $f(n,n) = 1$. For any other value of n, at least one game must be played. He wins or loses with probability 1/2. Therefore, by the law of total probability, we get:

$$f(k,n) = \frac{f(k-1,n)}{2} + \frac{f(k+1,n)}{2}$$

Suppose that $f(1,n) = m$.

Claim: $f(k,n) = km$

Proof: By strong induction on k. Clearly, the result is true for $k = 0, 1$. Suppose the result is true for k_0. Then $f(k_0,n) = k_0 m$, and $f(k_0-1,n) = (k_0-1)m$.

But we also know:

$$
\begin{aligned}
f(k_0, n) &= \frac{f(k_0-1,n)}{2} + \frac{f(k_0+1,n)}{2} \\
&= \frac{(k_0-1)m}{2} + \frac{f(k_0+1,n)}{2}
\end{aligned}
$$

So we have, by the induction hypothesis:

$$k_0 m = \frac{(k_0-1)m}{2} + \frac{f(k_0+1,n)}{2}$$

Therefore:

$$2k_0 m - (k_0-1)m = f(k_0+1,n)$$

So we get $f(k_0+1,n) = (k_0+1)m$, and the claim is proved.

From the claim, we have $1 = f(n,n) = nf(1,n)$. Therefore $f(1,n) = 1/n$.

2. Let Ω be a probability space, and $B_1, \ldots, B_n$ partition the space (i.e. the B_i are pairwise disjoint events whose union is Ω). Let A be any event in the space. Prove:

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A|B_i)\Pr(B_i)$$

Because the B_i partition Ω, we know that each $\omega \in \Omega$ is in exactly one B_i. We therefore have:

$$
\begin{aligned}
\Pr(A) &= \sum_{\omega \in A} \Pr(\omega) \\
&= \sum_{i=1}^{n} \sum_{\omega \in B_i : \omega \in A} \Pr(\omega) \\
&= \sum_{i=1}^{n} \Pr(A \cap B_i) \\
&= \sum_{i=1}^{n} \frac{\Pr(A \cap B_i)}{\Pr(B_i)} \Pr(B_i) \\
&= \sum_{i=1}^{n} \Pr(A|B_i)\Pr(B_i)
\end{aligned}
$$

Another solution arises from the fact that we know that $\Pr(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \Pr(A_i)$ if the A_i's are pairwise disjoint.

3. Suppose we flip n loaded coins, where each coin is heads with probability p. Then $\Omega = H, T^n$. If $A \subset [n]$, let Δ_A be the set of outcomes where all coins in A come up heads. Show that if $A \cap B = \emptyset$, then Δ_A and Δ_B are independent.

We have,

$$\begin{aligned}
\Pr(\Delta_A) &= \frac{1}{2^{|A|}} \\
\Pr(\Delta_B) &= \frac{1}{2^{|B|}} \\
\Pr(\Delta_A \cap \Delta_B) &= \frac{1}{2^{|A|+|B|}}
\end{aligned}$$

The third line is because A and B are disjoint.
Therefore:

$$\begin{aligned}
\Pr(\Delta_A|\Delta_B) &= \frac{\Delta_A \cap \Delta_B}{\Delta_B} \\
&= \frac{\frac{1}{2^{|A|+|B|}}}{\frac{1}{2^{|B|}}} \\
&= \frac{1}{2^{|A|}} \\
&= \Pr(\Delta_A)
\end{aligned}$$

4. Suppose you run an experiment. Each time you run it, there is a p probability of success, a q probability of failure, and a r probability of a "neutral" outcome. Suppose you run the experiment until you either fail or succeed. What is the probability that the last trial will be a success?

Note that we assume that $p+q+r=1$.
The probability of success if exactly n tosses are needed is:

$$r^{n-1}p$$

Therefore, the total probability of success is:

$$\sum_{n=1}^{\infty} r^{n-1}p = \frac{p}{1-r} = \frac{p}{p+q}$$

5. Consider the game of craps. You roll a pair of dice. If you roll a 2, 3, or 12, you lose. If you roll a 7 or 11, you win. If you roll someting else, the number x that you rolled is called your "point".

If you have a "point", your objective is then to roll x again before you roll a 7. You keep rolling this pair of dice until you either get another x or a 7. You win if you roll x before getting a 7, and you lose otherwise. What is the probability that you will win? Hint: problem 4 will come in very handy.

We use the law of total probability. Let A_k be the event that k was the sum of the two dice on the first roll. Let W be the winning event.

Then $P(W|A_7 \cup A_11) = 1$, and $P(W|A_2 \cup A_3 \cup A_12) = 0$.

We get, from problem 4 and the law of total probability:

$$\Pr(W) = \frac{\frac{3}{36}}{\frac{9}{36}}\frac{3}{36} + \frac{\frac{4}{36}}{\frac{10}{36}}\frac{4}{36} + \frac{\frac{5}{36}}{\frac{11}{36}}\frac{5}{36} + \frac{6}{36} + \frac{\frac{5}{36}}{\frac{11}{36}}\frac{5}{36} + \frac{\frac{4}{36}}{\frac{10}{36}}\frac{4}{36} + \frac{\frac{3}{36}}{\frac{9}{36}}\frac{3}{36} + + \frac{2}{36} \tag{1}$$

This is slightly less than 50%.

6. Suppose now there are three dice. Before the dice are rolled, you pick a certain number x from 1 to 6. For each x that is showing after the dice are rolled, you win one dollar. If no x shows you lose a dollar. Is this a good bet for you to make? Justify your answer.

Let X be the net win/loss after playing a game. There is 1 way out of 216 to get 3 x's, 15 ways of 216 to get 2 x's, and 75 ways of 216 to get 1 x. Thus, 91 of the 216 rolls give you at least one x. The remaining 125 rolls give you no x at all. So we get:

$$\begin{aligned} E(X) &= 3 \cdot \frac{1}{216} + 2 \cdot \frac{15}{216} + 1 \cdot \frac{75}{216} + (-1) \cdot \frac{125}{216} \\ &= \frac{108 - 125}{216} \\ &< 0 \end{aligned}$$

Therefore this is not a good bet.

7. I have three shells. I randomly place a ball under one of these three shells. I know where the ball is, but you do not. You pick a shell, and if the shell you pick covers the ball you win. Let's say you pick a shell. Then I remove a shell (other than the one that you picked) which I know does not have the ball under it. Then there are two shells remaining, and your ball must be under one of those two, one of which is the shell you picked. What is your probability of winning?

The probability is 1/3. Even though the ball must be under one of the two remaining shells, the shells themselves are not randomly chosen, but are rather chosen by me with knowledge of where the ball is – in other words, the shell that I am revealing to you is "biased". I can always remove an empty shell, so my doing so does not revise how likely it was for the ball to have been where it was to begin win.