# 21-228 Homework 3

Due September 25, 2001

**1. Remember that we have defined a relation to be a special kind of set. For (a) or (b) either give a proof for a "yes" answer, or a counterexample for a "no" answer (remember the counterexample must be justified too).**

**(a) Is the intersection of two equivalence relations on the same set an equivalence relation?**

This is true. Note that if $R_1$ and $R_2$ are equivalence relations on a set $A$, and if we let $R = R_1 \cap R_2$, then we have that $xRy$ iff $xR_1y$ and $xR_2y$. We now need to check the three properties:

Reflexivity: Since $R_1$ and $R_2$ are equivalence relations, we have $xR_1x$ and $xR_2x$, for each $x \in A$. Therefore, $xRx$ for each $x \in A$.

Symmetry: Let $x, y \in A$ be given so that $xRy$. Then $xR_1y$ and $xR_2y$. Therefore, since $R_1$ and $R_2$ are equivalence relations, it follows that $yR_1x$ and $yR_2x$. Therefore, $yRx$, and we have proved symmetry.

Transitivity: Let $x, y, z \in A$ be given so that $xRy$ and $yRz$. It then follows that $xR_1y$, $xR_2y$, $yR_1z$, and $yR_2z$. Since $R_1$ and $R_2$ are equivalence relations, it follows that $xR_1z$ and $xR_2z$. Therefore, $xRz$, and transitivity has been proved.

**(b) Is the union of two equivalence relations on the same set an equivalence relation?**

This is false. If $R_1$ and $R_2$ are equivalence relations on $A$, with $R = R_1 \cup R_2$, then we have $xRy$ iff $xR_1y$ *or* $xR_2y$. It turns out that $R$ will satisfy both the reflexive and symmetric requirements but may fail the transitivity requirement. A counterexample is if we let $A$ be the set of natural numbers. Then let $R_1$ be the relation $xR_1y$ iff $2|(x-y)$, and $R_2$ be the relation $xR_2y$ iff $3|(x-y)$. We have seen in class that these are both equivalence relations. We then know that $2R_14$, and $4R_27$. So $2R4$, and $4R7$. But since $2R_17$ and $2R_27$ are both false, it follows that $2R7$ cannot hold either. Thus transitivity fails.

**2. Show that the number of ways of obtaining the integer $k$ as a sum of a list of $n$ nonnegative integers is $\binom{n+k-1}{k}$. What if we require all the integers to be positive?**

If we consider the $n$ integers to be bookshelves, and the integer $k$ to represent $k$ books to be placed in these bookshelves, then this problem represents the number of ways to distribute $k$ identical books throughout $n$ bookshelves. In the second case, we require that each bookshelf be given at least 1 book, so we first give each bookshelf that one book, then there are $\binom{k-1}{k-n}$ ways to distribute the remaining $k-n$ books to the $n$ bookshelves.

**3. In how many ways can $n$ identical chemistry books, $r$ identical mathematics books, $s$ identical physics books, and $t$ identical astronomy books be placed on $k$ bookshelves?**

We consider this problem by first pretenting that all the books are distinct (for example, we assign each one a number within its group). As seen from class, the number of ways of doing this is $\frac{(k+n+r+s+t-1)!}{(k-1)!}$. Now, to make this equal to the actual number of arrangements, we notice that any two arrangements formed by rearranging books on the same subject are indistinguishable. Therefore, we dvide by $n!r!s!t!$, and we get:

$$\frac{(k+n+r+s+t-1)!}{(k-1)!n!r!s!t!} = \binom{k+n+r+s+t-1}{(k-1), n, r, s, t}$$

**4. In how many ways can 100 distinct beads be used to make three necklaces with 20 beads and four necklaces with 10 beads?**

By Theorem 1.6 on P.77 of Bogart, 3rd edition, there are $\frac{100!}{(20!)^3(10!)^4 3!4!}$ ways of partitioning the beads into three groups of 20 and 4 groups of 10. Once we have partitioned the set into such groups, there are $19!/2$ ways of arranging the beads in the groups of 20 to form a necklace, and $9!/2$ ways of arranging the beads in the groups of 10 to form a necklace. Since there are 3 groups of 20 and four groups of 10, we get a final answer of:

$$\frac{100!}{(20!)^3(10!)^4 3!4!}\left(\frac{19!}{2}\right)^3\left(\frac{9!}{2}^4\right)$$

**For 5 and 6, you may use the following fact without proof:**

**Fact 1: For Nim, the end is where there are no chips left in any stack. Suppose that there is a property $P$ satisfied by a position where all chips have been removed. Suppose further that:**

**a) Any legal move from any situation with property $P$ sends the game to a situation where property $P$ does not hold b) From any situation that does *not* satisfy property $P$, there is a legal move to a situation that *does* satisfy property $P$.**

**Then: If a player has to make a move from a state with property $P$, then her adversary has a winning strategy. Similarly, if a player moves from a state for which property $P$ does *not* hold, then that player has a guaranteed win.**

**5. The game of Nim is played as follows. We have $n$ stacks of chips, where stack $i$ starts with $x_i$ chips. Each player plays in turn, where in a turn, a player may remove any number of the chips from one pile, but may not remove chips from more than one pile. Whoever removes the last chip wins. Clearly, if there is only one pile, the first player is guaranteed to win. Suppose there are two piles, with sizes $x_1$ and $x_2$, where $x_1$ and $x_2$ are both positive (i.e. nonzero). For which values of $x_1$ and $x_2$ is the first player guaranteed a win? For which values is the second player guaranteed a win? Prove your answers. Note that *every* pair of positive integers should be in one of these two classes.**

The first player has a winning strategy iff $x_1 \neq x_2$, and the second player has a winning strategy if $x_1 = x_2$. To prove this, we appeal to the fact, where $P$ is the property "The two piles have the same number of chips". We must show that a) and b) hold in the statement of the fact.

a) is true because we must remove at least one chip from one pile, and only one pile may be touched. Therefore, if the two piles have the same number of chips before the move, the pile that has a chip removed must be smaller after the move, so property P can no loger hold.

b) is true because if the sizes $a_1$ and $a_2$ of the piles are different, without loss of generality we may assume that $a_1 > a_2$, so we may remove $a_1 - a_2 > 0$ chips from pile 1 to make the two piles equal.

So, when $x_1 = x_2$, the first player has to make a move when property $P$, so the second player has a winning strategy. Otherwise, the first player can force the second player to make a move from a position with property $P$, so the first player has a winning strategy.

**6. Again, we're playing the game of Nim described in Problem 5. Suppose now that there we start with $n$ stacks – under which conditions does the first player have a win, and what is the winning strategy in general? This analysis is harder (but there are parallels in the answers) than with only two stacks.**

At any point in the game, let $a_1, \ldots, a_n$ be the sizes of the $n$ piles. Consider their binary expansion, and the *nim-sum* as described in the hints. Then the first player wins iff this *nim-sum* has at least one non-zero bit, and the second player wins otherwise. So let property P be "the nim-sum of this position is zero". Then we need only show that a) and b) hold.

Property a) holds because we must change at least one bit in one pile, and are not allowed to change bits in any other pile. Therefore, we must change a bit in the sum.

Property b) holds as follows – investigate the left-most 1 in the nim-sum. There must be a corresponding 1 in one of the piles above. Pick a pile with such a corresponding 1, switch that 1 to a 0 in the binary representation of the pile. For the other 1's in the nim-sum (all of which must be to the right of that original 1), flip the bit in the corresponding position of the pile that we chose. The resulting num-sum is zero, and since all of the other modifications occur in less-significant bits than the one in which we changed a 1 to a 0, we have reduced the binary number for that pile, which therefore corresponds to a legal move.

You know, it's a lot easier if I describe this in person, as I'm too lazy to draw nice aligned tables in TeX.

**7. Chomp is a game played as follows. We start with an $m \times n$ grid of squares, where the lowerleft corner is the square $(1, 1)$, and the upper-right corner is the square $(m, n)$. Players alternate moving as follows: Each player, in her turn, must eat a square, and in doing so, also eats any square that is above and to the right. So, if a player *chomps* at $(x_0, y_0)$, she removes all $(x, y)$ with $x \geq x_0$ and $y \geq y_0$. The square at $(1, 1)$ is poisoned though – chomping it results in a loss. Give a strategy-stealing argument to show that the second player cannot have a winning strategy.**

**Aside: This implies that the first player has a winning strategy. However, your proof probably will give no hint as to the idea of the first player's winning move. Indeed, we know that the first player has a winning strategy in this case, but it is still unknown what that strategy is!**

If the second player has a winning strategy, it must work no matter what the first player does on her first move. Suppose that the first player removes only the upper-right corner $(m, n)$. Suppose that in this case, the second player's winning move would be to remove the square $(x_0, y_0)$. Then the resulting board, which would be a losing board for the first player, would consist of the original board, minus squares $(x, y)$ where $x \geq x_0$ and $y \geq y_0$. However, the first player, by chomping the square $(x, y)$ would present the second player with that exact same board – which means that the second player would be in a losing position. So both players would have a winning strategy, which is impossible.