21-228 Homework 3

Due September 25, 2001

1. Remember that we have defined a relation to be a special kind of set. For (a) or (b) either give a proof for a "yes" answer, or a counterexample for a "no" answer (remember the counterexample must be justified too).

(a) Is the intersection of two equivalence relations on the same set an equivalence relation?

(b) Is the union of two equivalence relations on the same set an equivalence relation?

2. Show that the number of ways of obtaining the integer k as a sum of a list of n nonnegative integers is $\binom{n+k-1}{k}$. What if we require all the integers to be positive?

3. In how many ways can n identical chemistry books, r identical mathematics books, s identical physics books, and t identical astronomy books be placed on k bookshelves?

4. In how many ways can 100 distinct beads be used to make three necklaces with 20 beads and four necklaces with 10 beads?

For 5 and 6, you may use the following fact without proof:

Fact 1: For Nim, the end is where there are no chips left in any stack. Suppose that there is a property P satisfied by a position where all chips have been removed. Suppose further that:

a) Any legal move from any situation with property P sends the game to a situation where property P does not hold b) From any situation that does *not* satisfy property P, there is a legal move to a situation that *does* satisfy property P.

Then: If a player has to make a move from a state with property P, then her adversary has a winning strategy. Similarly, if a player moves from a state for which property P does *not* hold, then that player has a guaranteed win.

5. The game of Nim is played as follows. We have n stacks of chips, where stack i starts with x_i chips. Each player plays in turn, where in a turn, a player may remove any number of the chips from one pile, but may not remove chips from more than one pile. Whoever removes the first chip wins. Clearly, if there is only one pile, the first player is guaranteed to win. Suppose there are two piles, with sizes x_1 and x_2, where x_1 and x_2 are both positive (i.e. nonzero). For which values of x_1 and x_2 is the first player guaranteed a win? For which values is the second player guaranteed a win? Prove your answers. Note that *every* pair of positive integers should be in one of these two classes.

6. Again, we're playing the game of Nim described in Problem 5. Suppose now that there we start with n stacks – under which conditions does the first player have a win, and what is the winning strategy in general? This analysis is harder (but there are parallels in the answers) than with only two stacks.

7. Chomp is a game played as follows. We start with an $m \times n$ grid of squares, where the lowerleft corner is the square $(1, 1)$, and the upper-right corner is the square (m, n). Players alternate moving as follows: Each player, in her turn, must eat a square, and in doing so, also eats any square that is either above or to the right. So, if a player *chomps* at (x_0, y_0), she removes all (x, y) with $x \geq x_0$ and $y \geq y_0$. The square at $(1, 1)$ is poisoned though – chomping it results in a loss. Give a strategy-stealing argument to show that the second player cannot have a winning strategy.

Aside: This implies that the first player has a winning strategy. However, your proof probably will give no hint as to the idea of the first player's winning move. Indeed, we know that the first player has a winning strategy in this case, but it is still unknown what that strategy is!