21-228 Homework 2

Due September 18, 2001

1. Without appealing to the formula for $\binom{n}{k}$, prove $\binom{n}{m}\binom{n-m}{k} = \binom{n}{k}\binom{n-k}{m}$.

Think in terms of elements and sets. Let A be a set of size n. We want to choose a set B of size m and C of size k. To do so, we may either choose the elemetns fo B first and then the elements of C. The value on the left hand side represents the number of ways to do so, by selecting the elements of B first and then the elements of C. The value on the right hand side represents the number of ways to do so by first choosing the elements of C, and then the elements of B. Since both methods accomplish the same task, the two quantities must be the same.

2. Prove the formula:

$$\sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k},$$

again without using the formula for a binomial coefficient.

The value on the right-hand side represents the number of ways to choose k elements from a set of $m+n$. Let A be a set of size $m+n$, so that $A = \{a_1, \ldots, a_{m+n}\}$. Let $B = \{a_1, \ldots, a_m\}$ and let $C = \{a_{m+1}, \ldots, a_{m+n}$. Then B has size m and C has size n. Now, we may choose k elements from A as follows:

For any j from 0 to k, we may choose j elements to be from B, and $k-j$ to be from C. For each possible j, there are $\binom{m}{j}$ ways to pick the elements from B, and $\binom{n}{k-j}$ ways to pick the elements from C. Therefore, by the product principle, the total number of ways to carry out this procedure for a given j is $\binom{m}{j}\binom{n}{k-j}$. Summing this over all possible j gives us our answer.

3. What value of k makes $\binom{n}{k}$ maximized for a given value of n? Prove your answer. To do this, consider the relationship between $\binom{n}{k}$ and $\binom{n}{k+1}$ and figure out in which cases the former is larger than the latter.

Let's fix a value of n and k. Then we have:

$$\begin{aligned} \frac{\binom{n}{k}}{\binom{n}{k+1}} &= \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k+1)!(n-k-1)!}} \\ &= \frac{(k+1)!(n-k-1)!}{k!(n-k)!} \\ &= \frac{(k+1)}{(n-k)} \end{aligned}$$

This value is greater than one if and only if $k+1 > n-k$. Therefore, for a given n, $\binom{n}{k}$ is maximized at the largest value of k so that $k+1 \leq n-k$ holds. This happens when $2k+1 \leq n$. So, $\binom{n}{k}$ is maximized when k is the integer nearest to $n/2$.

4. Show that:

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

This is a direct result of VanderMonde's formula.

5. A person wants to walk from a certain point in a city to a point seven blocks north and eight blocks east. In how many ways may she go from one point to the other and walk exactly 17 blocks? In how many ways may she go from one point to the other and walk exactly 16 blocks?

For the first question, notice that we are instead asking for the number of sequences that have exactly 17 elements that result in us moving exactly 7 north and 8 east. For this to occur, there must be exactly one south step or exactly one west step. This is true because there are 15 blocks needed to make a direct route from start to finish, and if we make even one out-of-the-way step, we need now 16 more to get the destination.

If there is exactly one south step, then the number of paths is the same as the number of lists of 17 elements containing one "S", 8 "N"s, and 8 "E"s. From the multinomial theorem, this works out to $\binom{17}{8,8,1} = 17 * \binom{16}{8} = \frac{17!}{8!8!}$.

If there is exactly one west step, then the number of paths is the same as the number of lists of 17 elements containing one "W", 7 "N"s and 9 "E"s. Using the multinomial theorem, we get $\binom{17}{9,7,1} = 17 * \binom{16}{9} = \frac{17!}{7!9!}$.

For the second question, suppose we start at $(0, 0)$, and want to end at $(7, 8)$. We are starting at an "even" position – that is, the sum of the coordinates is even. And we wish to end at an "odd" position – that is, the sum of the coordinates is odd. Each step we take increases or decreases one of the coordinates by exactly 1 – therefore, if we are at an "odd" position we go to an "even" position, and from an "even" position we go to an "odd" position. So, moving an odd number of steps from an "even" position will always take us to an "odd" position, and moving an even number of steps from an "even" position will take us to an "odd" position. Thus, we can never get from $(0, 0)$ to $(7, 8)$ using an even number of steps.

6. Suppose we have a standard deck of playing cards, and we use only the clubs. How many ways can we, to four people, distribute the 13 cards so that each person gets at least 3 of them? What is the total number of ways to distribute the 13 cards? This gives us a rough (but not quite right) idea of how likely it is that if we deal out all 52 cards to the four people, with each person getting 13 of them, that each one will get at least three clubs.

For the first answer, there are $\binom{13}{3}$ combinations of cards for the first player, $\binom{10}{3}$ for the next, $\binom{7}{3}$ for the next, and then the remaining player must get four cards. However, any of the four players could be the "lucky" one to get the four-club hand, so we must multiply by 4. Therefore, we get

$$4\binom{13}{3}\binom{10}{3}\binom{7}{3} = 4\binom{13}{4,3,3,3}$$

The total number of distributions is 4^{13} – this the same as the number of functions from the set of 13 cards to the set of 4 people.

7. Show that:

$$\sum_{k=0}^{n} \frac{2n!}{k!^2(n-k)!^2} = \binom{2n}{n}^2$$

Squaring both sides of the result of problem 4 gives us

$$\left(\sum_{i=0}^{n} \binom{n}{i}^2\right)^2 = \binom{2n}{n}^2$$

Expanding the left hand side gives:

$$\sum_{i=0}^{n}\sum_{k=0}^{n} \binom{n}{i}^2 \binom{n}{k}^2 = \binom{2n}{n}^2$$

Switching the order of the sum signs above gives:

$$\sum_{k=0}^{n}\sum_{i=0}^{n} \binom{n}{i}^2 \binom{n}{k}^2 = \binom{2n}{n}^2$$

Now, since k is constant in the inner sum we can pull the $\binom{n}{k}^2$ part to the outer sum, yielding:

$$\sum_{k=0}^{n} \left(\binom{n}{k}^2 \sum_{i=0}^{n} \binom{n}{i}^2\right) = \binom{2n}{n}^2$$

Now, apply the result of problem 4 to the inner sum to get:

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n}{n} = \binom{2n}{n}^2$$

Now, we use the formula for binomial coefficients on the left hand side to get:

$$\sum_{k=0}^{n} \frac{(n!)^2}{k!^2(n-k)!^2} \frac{(2n)!}{n!n!} = \binom{2n}{n}^2$$

Cancelling on the left hand side gives us:

$$\sum_{k=0}^{n} \frac{(2n)!}{k!^2(n-k)!^2} = \binom{2n}{n}^2$$