21-228 Homework 1

Due Tuesday, September 11

All answers must be justified to receive credit.

1. How many ways can you distribute three different pieces of candy to five children if no child can receive more than one piece? What about if any child can receive any number?

For the first question, notice that we can give the first piece of candy to any of the five children. The second piece of candy may be given to any of four children. The third piece has three possibilities. Then, by the *Generalized* product principle, there are $5 * 4 * 3 = 60$ possibilities, as this corresponds to lists of three distinct children of the five.

For the second question, we no longer require distinctness, so we have a total of 5 possibilities for each piece of candy. Therefore, there are $5 * 5 * 5 = 125$ possibilities.

2. A coffee machine allows you to have your coffee either plain, or with single or double portions of sugar and/or cream. In how many ways can you choose your coffee?

Each possible style of coffee corresponds to a list whose first entry is the "amount" of sugar, and whose second is the "amount" of cream. There are three possible entries for each entry of the list (none, single portion, or double portion), and two entries, so by the product principle, there are $3 * 3 = 9$ possibilities.

3. For any two sets A and B, we define the set $A - B$ to be the set of all elements of A that are not in B. Suppose now that every element of Y is also an element of X. Use the sum principle to prove that $|X - Y| = |X| - |Y|$.

By definition, we know that Y and $X - Y$ are disjoint. Furthermore, we know that $Y \cup (X - Y) = X \cup Y$. Since each element of Y is also an element of X, we know that $X \cup Y = X$. Therefore, $Y \cup (X - Y) = X$. Therefore, we have, by the sum principle, that $|Y| + |X - Y| = |X|$. Thus, $|X| - |Y| = |X - Y|$.

4. Using induction, prove the generalized product principle.

Base case: $k = 1$. Here, it is clear that there are only m_1 possible lists.

Inductive step. Suppowe we know the result to be true for $k = k_0$. Then consider lists with $k_0 + 1$ entries. We know from the inductive hypothesis, that there are always $\prod_{i=1}^{k_0} m_i$ ways of generating the first k_0 entries of the list. From the hypothesis of the generalized product principle, there are, no matter how we choose these first k_0 entries, m_{k_0+1} ways of choosing the $k_0 + 1$st entry. From the regular product principle, then we have $\left(\prod_{i=1}^{k_0} m_i\right) \cdot m_{k_0+1}$ possibilities.

5. Let A and B be sets where A has n elements and B has k elements.

Here, give only brief explanations, not detailed proofs.

(a) Explain why the number of functions from A to B is the same as the number of ways of listing n elements from a set of k elements, with repetition allowed.

If we order the set $A = \{a_1, \ldots, a_n\}$, then we can let $f(a_i)$ be the ith entry in the list. Thus, each function from A to B corresponds to a listing of n elements from a set of k.

(b) Explain why the number of *one-to-one* functions from A to B is the same as the number of ways of listing n elements from a set of k elements, *without* repetition allowed.

In this case, we again list, but since the function is one-to-one, no each possible value from B can occur at most once in the list as constructed in part a). Therefore, we have a listing without repetition of n elements from a set of k.

6. Use the extended pigeonhole principle to show that for any set of seven integers at least three of the nonnegative differences between them have the same last digit.

There are a total of $\binom{7}{2} = 21$ possible such differences. There are a total of only 10 possible last digits. Thus, by applying the extended pigeonhole principle with $m = k = 2$, we see that there are at least three such differences that have the same last digit.