21-228 Exam 3

Name:______________________________________

December 10, 2001

- Place your name and Section letter (whichever section you are *sitting in on*) on the space provided.
- You have 50 minutes. Pace yourself appropriately.
- **All answers must be justified to receive credit**. Please write as legibly as possible.
- You may use the back sides (and any other spare space on the exam pages) if you wish. Clearly indicate where your work is if you wish to do so – we will not be responsible for grading such work otherwise.
- The back page contains a formula sheet, which you may tear off if desired.
- All graphs in your answers should be simple – that is, no loops or parallel edges.
- Good luck!

Question	Score	Possible
1		80
2		20
Total		100

1. This problem refers to the graph on the following page.

Answer these questions on the following few pages.

(a) (10 points) Suppose that we have a matrix A so that the entry in row i and column j is the distance of a direct road from i to j. How do we know if the resulting data can be represented as an undirected graph? (Give a bief answer – no more than one sentence).

If A is symmetric, or, if, equivalently, if $A_{ij} = A_{ji}$ for all i, j.

(b) (20 points) If we were to use Kruskal's algorithm to form a minimum spanning tree of this graph, in which order would the edges be entered?

Below is a listing of when each edge is considered, followed by whether it gets put in the tree.

1. v_1v_3, yes
2. v_2v_4, yes
3. v_5v_7, yes
4. v_2v_5, yes
5. v_4v_7, no – forms cycle by v_4, v_2, v_5, v_7.
6. v_4v_5, no – forms v_4, v_5, v_2 cycle
7. v_3v_4, yes
8. v_1v_4, no – forms $v - 1, v_3, v_4$ cycle
9. v_1v_2, no – forms v_1, v_2, v_5, v_4, v_3 cycle.
10. v_4v_6, yes

We may stop here because the edges entered now form a tree.

(c) (20 points) If we were to use Prim's algorithm, starting at vertex 1, now in which order would the edges be added?

1. v_1v_3
2. v_3v_4
3. v_2v_4
4. v_2v_5
5. v_5v_7
6. v_4v_6

(d) (20 points) If we were to run Dijkstra's algorithm on this graph, finding the shortest path from vertex 1 to all other vertices, which are the first two vertices that would be added to S (after vertex 1 itself)? Hint: you do *not* have to run the algorithm to completion to answer the question.

After the first iteration we see that $t_1 = 0$, $t_2 = 9$, $t_4 = 8$, and $t_3 = 1$. So, the next vertex to get added is v_3.

After the next iteration, considering the edges leaving v_3, we see that $t_4 = 1 + 7 = 8$, $t_6 = 1 + 11 = 12$, and $t_4 = \min(8, 1 + 7) = 8$. Therefore, v_4 is the next vertex to get added.

(e) (10 points) How many colors are needed to color the vertices of the graph so that no two adjacent vertices have the same color? Why can't we accomplish this with fewer colors? Give a reason that is easily generalizable to a large, interesting class of graphs.

Three colors are needed. This is accomplished by coloring the central vertex one color, and the outer vertices alternating colors.

Fewer colors are not possible because the graph is not bipartite, and only bipartite graphs are 2-colorable.

2. (20 points) Let G be a graph. A *perfect matching* on G is a matching M that saturates all vertices of the graph. Show that if G has a perfect matching, then every vertex cover has size at least $|V|/2$.

A perfect matching sets the edges in pairs. In this manner, $|V||/2$ pairs are needed. So, M will have $|V||/2$ edges, and therefore any cover must, by Formula 5, have at least this many vertices. Note that bipartiteness is completely irrelevant.

Some other stuff

1. A graph is *bipartite* if its vertex set can be partitioned into two sets A, B so that each edge has one end in A and the other in B
2. A graph is bipartite iff it has no cycles of odd length.
3. A *Matching* is a set of edges, no two of which share an end.
4. A *Vertex Cover* is a set S of vertices, so that every edge in G has at least one end in S.
5. If G is a graph, and M a matching, and S a vertex cover, then $|S| \geq |M|$.
6. A matching M *saturates* a set S of vertices iff each vertex in S is an end of some edge in M.
7. If G is a bipartite graph, then the size of a maximum matching is the same as the size of a minimum vertex cover.
8. A *matroid* on a set X is a collection $\mathcal{F}$ of subsets of X so that:
   (a) If $A \in \mathcal{F}$, and $B \subseteq A$, then $B \in \mathcal{F}$.
   (b) If $|A| < |B|$, and $A, B \in \mathcal{F}$, then there is some $x \in B - A$ so that $A \cup \{x\} \in \mathcal{F}$.
9. If $\mathcal{F}$ is a matroid in X, then $A \in \mathcal{F}$ is a *basis* if no element of $\mathcal{F}$ properly contains A. That is, A cannot be extended to another element of $\mathcal{F}$ by adding another element.