21-228 Exam 1

Name:____________________________________

October 3, 2001

- Place your name and Section letter (whichever section you are *sitting in on*) on the space provided.
- You have 50 minutes. Pace yourself appropriately.
- Answers may be left in terms of factorials and exponents, but not binomial/multinomial coefficients, Stirling numbers, or other "higher level" forms.
- **All answers must be justified to receive credit**. Please write as legibly as possible.
- You may use the back sides (and any other spare space on the exam pages) if you wish. Clearly indicate where your work is if you wish to do so – we will not be responsible for grading such work otherwise.
- The back page contains a formula sheet, which you may tear off if desired.
- Good luck!

Question	Score	Possible
1		20
2		20
3		20
4		20
5		20
Total		100

1. (20 points)

A 10 digit number is an integer with 10 digits, the first of which is *nonzero*

(a) (10 points) What is the number of 10 digit numbers with no two successive digits equal?

There are 9 choices for each digit, and 10 choices, therefore, there are a total of 9^10 such numbers.

(b) (10 points) How many 10 digit numbers have at least one pair of successive digits equal?

There are $(9 * 10^9)$ total 10 digit numbers. Subtracting part (a) from this result gives $9 * 10^9 - 9^10$.

2. (20 points) Give a *combinatorial* proof of the following formula:

$$\binom{n}{m}\binom{m}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}\binom{n-k}{m-k}$$

That is, do not use any formulas that we have learned in class. Work directly from what $\binom{n}{k}$ means in terms of being the number of k-element subsets of an n-element set. You may assume that n, m, k, j are nonnegative integers with $n \geq m \geq k \geq j$.

The lefthand side represents the number of ways of choosing m army members, then k officers out of these m, then choosing j generals out of these k officers.

The righthand side represents the number of ways of choosing j generals, then, $k - j$ officers to *not* be generals, then $m - k$ soldiers to *not* be officers.

3. (a) (10 points) How many ordered lists of five *positive* integers (i.e. lists of the form $(x_1, x_2, \ldots, x_5)$) are there whose elements sum to 15? (i.e. $x_1 + \cdots + x_5 = 15$)?

This is the same as arranging 15 identical books on 5 distinct bookshelves, so that each shelf gets at least one book. Thus, we place one book on each shelf, leaving 10. By formula, we get $\binom{14}{10}$. or, $\frac{14!}{10!4!}$.

(b) How many ways can 100 distinct pieces of candy be placed into five identical bags, so that three of the bags contain 30 pieces each, and the remaining two contain 5 pieces each?

By the formula, this is

$$\frac{100!}{(30!)^3(5!)^2(3!)(2!)}$$

4. (a) (10 points) How many *surjections* are there from a 20 element set *onto* a 15 element set? You may leave your answer in terms of Stirling numbers, but you must justify your procedure. Hint: If $f : A \to B$ is a surjection, think about the collection of sets $f^{-1}(\{b\}) = \{a \in A : f(a) = b\}$.

The stirling number $S(20, 15)$ is the number of ways of arranging 20 elements into 15 indistinguishable nonempty classes. Each surjection defines a way of arranging 20 elements into 15 *distinguishable* nonempty classes. Therefore, the answer is $S(20, 15)15!$.

(b) (10 points) A *derangment* of a set A is a *bijection* $f : A \to A$ so that for *all* $a \in A$, $f(a) \neq a$. How many derangments of A are there if A has 5 elements?

There are 120 mappings total.

Of these, for each single element $a \in A$, there are 24 mapping fixing a, and 5 elements that this applies to.

There are, then 6 mappings fixing any given pair of elements in A, and there are 10 such pairs.

There are 2 mappings fixing any given triplet, and there are 10 triplets.

There is 1 mapping fixing any given set of four elements, and there are 5 such sets.

Then there is 1 mapping fixing all 5.

By Inclusion/exclusion, therefore, we get

$$120 - 5 * 24 + 6 * 10 - 2 * 10 + 1 * 5 - 1 = 44$$

5. Suppose we are on the three-dimensional coordinate system at the point $(0, 0, 0)$. In one step, we may move one unit in any *one* of the three coordinates (so we can go to $(\pm 1, 0, 0)$, or $(0, \pm 1, 0)$, or $(0, 0, \pm 1)$ in the first step). The coordinate system extends infinitely far in all directions, and we may use as much space as needed.

(a) (10 points) How many ways can we get from $(0, 0, 0)$ to $(8, 3, 5)$ using exactly 18 steps?

Out of the 18 steps, we have exactly enough steps to make one "misstep". The misstep can either be in the x, y, or z, coordinate. In the first case, there are $18!/(9!3!5!1!)$ possible routes. The second and third cases are similar, and we get

$$\frac{18!}{9!3!5!1!} + \frac{18!}{8!4!5!1!} + \frac{18!}{8!3!6!1!}$$

(b) (10 points) How many ways can we get from $(0, 0, 0)$ to $(8, 3, 5)$ using exactly 21 steps?

This is impossible, since each step takes you from a point with an odd-numbered sum of coordinates to an even-numbered-sum of coordinates, and conversely. Therefore, 21 steps from the origin will always take you to a point whose coordinates sum to an odd number, but $8 + 3 + 5$ is even.

1. If X and Y are disjoint, $|X \cup Y| = |X| + |Y|$
2. S is a set of lists of length k, there are m_1 different first elements of lists in S, for each way of specifying the first $i-1$ entries, there are m_i ways of choosing the next, then S has $m_1 \cdot m_2 \cdot \cdots \cdot m_k$ lists.
3. # lists of k elements out of n, repetition forbidden: $\frac{n!}{(n-k)!}$.
4. # lists of k elements out of n, repeition allowed: n^k.
5. # of k-element subsets of n-element set: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
6. n distinct chairs and m different colors of paint, and $k_1 + \cdots + k_m = n$, then there are $\binom{n}{k_1, \ldots, k_m} = \frac{n!}{k_1!k_2!\ldots k_m!}$ ways of painting the n chairs so that exactly k_i chairs are painted in color i.
7. # of partitions of an n element set into j_1 classes of size 1, j_2 classes of size 2, ... , j_n classes of size n: $\frac{n!}{\prod_{i=1}^{n}(i!)^{j_i} j_i!}$
8. Arrangements of k distinct books on n distinct bookshelves: $\frac{(n+k-1)!}{(n-1)!}$
9. If books are identical: $\binom{n+k-1}{k}$
10. # of paritions of an m-element set into n classes

$$S(m,n) = \frac{1}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m$$

11. There are $\binom{k}{n} \frac{(k-1)!}{(n-1)!}$ arrangements of k books into n nonempty stacks (two arrangements formed by rearranging entire stacks are the same arrangement).
12. A is a set of arrangmenets, P is a set of properties that arrangements in A may have. For $S \subseteq P$, $N_{\geq}(S)$ is the number of arrangmenents with at least the properties in S, and $N_{=}(S)$ is the number of arrangements with the properties in S and no others. Then if $S \subseteq P$:

$$N_{=}(S) = \sum_{J=S}^{P} (-1)^{|J|-|S|} N_{\geq}(J)$$