Department of Mathematical Sciences
Carnegie Mellon University Fall 2001

21-121 Calculus 1 (IM/Econ)

Handout #1:
Applications of Calculus to Business and Economics

Notation:

c	Total cost (typically given in terms of q)
$\frac{dc}{dq}$	Marginal cost
$\overline{c}$	Average cost
C	National consumption (typically given in terms of I)
$\frac{dC}{dI}$	Marginal propensity to consume
I	National income
p	Demand (also called price)
P	Profit
q	Number of units
r	Total revenue (typically given in terms of q)
$\frac{dr}{dq}$	Marginal revenue
S	National savings
$\frac{dS}{dI}$	Marginal propensity to save

Summary of Definitions:

A **demand function** $p = f(q)$ describes the relationship between the price p per unit of a product and the number q of units the consumers will buy at that price.

Example: Suppose that $p = 25 - 0.02q$. This relation tells us that in order to sell 200 units of product, the price must be

$$p|_{q=200} = 25 - 0.02(200) = 25 - 4 = 21$$

$21 per unit. Alternatively, the relation tells us that if the price is $11 per unit, then there will be a demand for

$$11 = 25 - 0.02q \Longrightarrow q = \frac{25 - 11}{0.02} = \frac{14}{0.02} = 700$$

700 units of product.

A **revenue function** $r = f(q)$ gives the relationship between the number of units sold and the total revenue received. Observe that the total revenue is equal to the price p per unit multiplied by the number q of units sold. In other words $r = pq$. So, if the demand function is given, then the revenue function can be found.

Example: Suppose again that $p = 25 - 0.02q$. Then the revenue function is

$$r = pq = (25 - 0.02q)q = 25q - 0.02q^2.$$

If 200 units of product are sold, then the total revenue received is

$$r\big|_{q=200} = 25(200) - 0.02(200)^2 = 5000 - 800 = 4200$$

\$4200.

The **marginal revenue** is defined as the instantaneous rate of change of the revenue r with respect to the number of units sold q. So

$$\text{marginal revenue} = \frac{dr}{dq}.$$

The marginal revenue can be interpreted as the approximate increase (decrease) in revenue from selling one more (less) unit of product.

Example: If $r = 25q - 0.02q^2$, then the marginal revenue is

$$\frac{dr}{dq} = \frac{d}{dq}\left[25q - 0.02q^2\right] = 25 - 0.04q.$$

When 200 units of product are sold the total revenue was found to be \$4200, and the marginal revenue is

$$\left.\frac{dr}{dq}\right|_{q=200} = 25 - 0.04(200) = 25 - 8 = 17.$$

Thus, if one more unit is sold, the total revenue will increase by approximately \$17, i.e. the total revenue received for selling 201 units is about \$4217.

The **average cost** $\overline{c} = f(q)$ tell us the average cost $\overline{c}$ per unit of producing q units.

Example: If $\overline{c} = 5 + \frac{1000}{q}$ and we wish to produce 200 units of product, then

$$\overline{c}\big|_{q=200} = 5 + \frac{1000}{(200)} = 5 + 5 = 10.$$

Thus to produce 200 units, it costs on average \$10 per unit.

A **cost function** $c = f(q)$ gives the total cost c of producing and marketing q units of product. Notice that the total cost is equal to the average cost $\overline{c}$ multiplied by the number of units q produced. So, if the average cost is given, then we can find the total cost function using the formula $c = \overline{c}q$.

Example: Again suppose that the average cost is given by $\overline{c} = 5 + \frac{1000}{q}$. The cost function is then

$$c = \overline{c}q = \left(5 + \frac{1000}{q}\right)q = 5q + 1000.$$

So the total cost of producing 200 units is

$$c\big|_{q=200} = 5(200) + 1000 = 2000$$

\$2000.

The **marginal cost** is the instantaneous rate of change of the cost c with respect to the number of units q produced. So

$$\text{marginal cost} = \frac{dc}{dq}.$$

The marginal cost provides an approximate increase (decrease) in total cost from producing one more (less) unit of product.

Example: If the cost function is given by $c = 5q + 1000$, then the marginal cost is

$$\frac{dc}{dq} = \frac{d}{dq}[5q + 1000] = 5.$$

When 200 units are produced, we found that the total cost was \$2000. The marginal cost is

$$\left.\frac{dc}{dq}\right|_{q=200} = 5.$$

So, if we wanted to produce one more unit, the total cost would increase by approximately \$5, i.e. the cost for production of 201 units is about \$2005.

A **profit function** $P = f(q)$ gives the profit P made when q units of product are produced and sold. If the revenue r and the cost c are known, then the profit is $P = r - c$. Note that a negative value for P represents a loss.

Example: Suppose that $r = 25q - 0.02q^2$ and $c = 5q + 1000$, then the profit is given by

$$P = r - c = \left(25q - 0.02q^2\right) - (5q + 1000) = -0.02q^2 + 20q - 1000.$$

If 200 units are produced and sold, then the profit made is

$$P|_{q=200} = -0.02(200)^2 + 20(200) - 1000 = -800 + 4000 - 1000 = 2200$$

\$2200.

A **consumption function** $C = f(I)$ expresses the total national consumption C in terms of the national income I. The **marginal propensity to consume** is the instantaneous rate of change of C with respect to I. Thus

$$\text{marginal propensity to consume} = \frac{dC}{dI}.$$

The marginal propensity to consume describes how quickly consumption changes with respect to income.

Similarly, a **savings function** $S = f(I)$ gives the total national savings S in terms of the national income I. Notice that whatever income not used to consume is saved; therefore $S = I - C$. The **marginal propensity to save** is the instantaneous rate of change of S with respect to I. That is

$$\text{marginal propensity to save} = \frac{dS}{dI}.$$

The marginal propensity to save describes how fast savings changes with respect to income.

Example:Suppose that the national consumption function is

$$C = \frac{5\left(2\sqrt{I^3}+3\right)}{I+10},$$

with I and C measured in billions of dollars. The national savings function is

$$S = I - C = I - \frac{5\left(2\sqrt{I^3}+3\right)}{I+10}.$$

If the national income is \$100 billion, then national consumption is

$$C|_{I=100} = \frac{5(2(1000)+3)}{110} = \frac{10015}{110} \approx 91.045$$

about \$94 billion. National savings is

$$S|_{I=100} = 100 - C|_{I=100} \approx 100 - 91.045 = 8.955$$

about \$6 billion. We also find that the marginal propensity to consume is

$$\begin{aligned}\frac{dC}{dI} &= \frac{d}{dI}\left[\frac{5\left(2\sqrt{I^3}+3\right)}{I+10}\right]\\ &= 5\left(\frac{(I+10)\frac{d}{dI}\left[2I^{\frac{3}{2}}+3\right] - \left(2I^{\frac{3}{2}}+3\right)\frac{d}{dI}[I+10]}{(I+10)^2}\right)\\ &= 5\left(\frac{\sqrt{I^3}+30\sqrt{I}-3}{(I+10)^2}\right),\end{aligned}$$

and the marginal propensity to save is

$$\frac{dS}{dI} = \frac{d}{dI}[I-C] = 1 - \frac{dC}{dI} = 1 - 5\left(\frac{\sqrt{I^3}+30\sqrt{I}-3}{(I+10)^2}\right).$$

Thus when $I = 100$, the marginal propensity to consume and save are respectively

$$\left.\frac{dC}{dI}\right|_{I=100} \approx 0.536 \quad \text{and} \quad \left.\frac{dS}{dI}\right|_{I=100} \approx 0.464.$$

If the national income were to increase from \$100 billion by \$1 billion, the nation would consume about \$536 million and save about \$464 million of that increase.

Reference:*Introductory Mathematical Analysis for Business, Economics, and Life Sciences*, Ernest F. Haeussler, Jr. & Richards S. Paul (Eighth Edition)