Concepts of Math: Recitation 15 Solutions

October 26, 2015

Combinatorics Formulas

1. In class we proved that the number of non-negative integer solutions $(x_1, x_2, \ldots, x_k)$ to

$$x_1 + x_2 + \ldots + x_k = n$$

is

$$\binom{n+k-1}{n}.$$

Count the positive integer solutions $(x_1, x_2, \ldots, x_k)$ to

$$x_1 + x_2 + \ldots + x_k = n.$$

Your answer should be a single binomial coefficient.

Solution: We model the positive integer solutions of $x_1 + x_2 + \ldots + x_k = n$ by placing n identical marbles into k different boxes, at least one marble per box. First place one marble in each of the k boxes, then distribute the rest $n - k$ marbles, any number per box. This can be done in $\binom{(n-k)+k-1}{n-k} = \binom{n-1}{n-k} = \binom{n-1}{k-1}$.

2. By counting a set it two ways, give a combinatorial proof of

$$n^2 = 2\binom{n}{2} + n.$$

Solution: We will prove this claim by counting in two ways. Using the rule of product, we see that the quantity on the left counts the number of 2-letter words that can be made with an n-letter alphabet.

We will show that the expression on the right counts the same quantity as well. To see this, we partition 2-letter words in to two types: those with distinct letters, and those with repeated letters. There are $2\binom{n}{2}$ words of the first type. Since there are $\binom{n}{2}$ ways to pick the letters, and 2 ways to order them, our claim follows from the rule of product.

There are also n words of the second type, as choosing the repeated letter uniquely defines the word. Therefore, by the rule of sum, the number of 2-letter words over an n-letter alphabet is $n^2 = 2\binom{n}{2} + n$.

Since we showed that both the expression on the right and the expression on the left count the same quantity, they must be equal.

3. By counting a set it two ways, give a combinatorial proof of

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + n.$$

Solution: We will prove this claim by counting in two ways. Using the rule of product, we see that the quantity on the left counts the number of 3-letter words that can be made with an n-letter alphabet.

We will show that the expression on the right counts the same quantity as well. To see this, we partition 3-letter words into three types: those with 3 distinct letters, and those that have 2 distinct letters, and those that have only the one type of letter.

There are $6\binom{n}{3}$ words of the first type. Since there are $\binom{n}{3}$ ways to pick the letters, and $3! = 6$ ways to order them, our claim follows from the rule of product.

There are $6\binom{n}{2}$ words of the second type. Note that in this case, one letter must be repeated exactly twice, and the other must not be repeated. Since there are $\binom{n}{2}$ ways to pick the two letters, 2 ways to pick which letter appears with multiplicity 2, and 3 ways to order the resulting letters ($3!/2 = 3$) our claim follows from the rule of product.

Finally, there are n words of the third type, as choosing the repeated letter uniquely defines the word. Therefore, by the rule of sum, the number of 3-letter words over an n-letter alphabet is $n^3 = 6\binom{n}{3} + 6\binom{n}{2} + n$.

Since we showed that both the expression on the right and the expression on the left count the same quantity, they must be equal.

4. By counting a set it two ways, give a combinatorial proof of

$$\binom{2n}{n} = 2\binom{2n-1}{n-1}.$$

Solution: We will prove this claim by counting in two ways. Note that by definition, the LHS counts the number of n-element subsets of a set of size $2n$.

To see that the RHS also counts this quantity, without loss of generality assume that the $2n$-element subset in question is $\{1, 2, \ldots, 2n\}$. Note that any n-element subset of this set either includes 1 as an element, or doesn't. There are $\binom{2n-1}{n-1}$ set of the first type as the n-element subset in question would be defined uniquely by which $n-1$ elements of $\{2, \ldots, 2n\}$ it includes. There are $\binom{2n-1}{n-1}$ set of the second type as well. Such a set is uniquely defined by the $n-1$ elements of $\{2, \ldots, 2n\}$ that don't belong to it. Therefore, by the rule of sum, the number of n elements subsets of a $2n$ element set is $2\binom{2n-1}{n-1}$ as claimed. This completes the proof.

5. Using the Manhattan walk interpretation of Pascal's triangle, give a combinatorial proof to

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

Solution: The proof is by double counting. We will show that both sides count the number of ways of creating a k-delegate committee with m distinguished superdelegates.

The LHS counts this quantity because we can we can first choose the k-delegates in $\binom{n}{k}$ ways, and then choose the m superdelegates from this k-person group in $\binom{k}{m}$ ways. The result follows from the rule of product.

The RHS counts this quantity because we can first choose the m superdelegates in $\binom{n}{m}$ ways, then choose the remaining $k-m$ delegates from the remaining $n-k$ people in $\binom{n-m}{k-m}$. Once again, the result follows from the rule of product. This completes our proof.

6. Three couples, the Smiths, Joneses, and Murphys, are going to form a line.

   (a) In how many such lines will Mr. and Mrs. Jones be next to each other?
   Solution: Superglue Mr. and Mrs. Jones together. We can do this in two ways: one so that Mr. Jones comes before Mrs. Jones, and one so that he comes after. Once the Joneses are glued together, we only need to arrange 5 items in a line, with no additional constraints. There are 5! ways of doing this. Factoring in the 2 ways the Joneses can be ordered, the final answer becomes $2 \cdot 5!$.

   (b) In how many such lines will Mr. and Mrs. Jones be next to each other and Mr. and Mrs. Murphy be next to each other?
   Solution: The idea is the same as above. Superglue the Joneses together and do the same to the Murphys. We are now permuting only 4 items in a line with no additional constraints. There are 4! ways of doing this. There are two ways of ordering the Joneses among themselves, and another 2 ways of doing the same for the Murphys. By the rule of product, the final answer is $4 \cdot 4!$.

   (c) In how many such lines will at least one couple be next to each other? Do not use inclusion-exclusion (we have not done that yet).
   Solution: We will count the number of permutations of these people that leaves no couples together, and subtract it from 6!, which is the total number of possible permutations.

   To count this quantity, we will instead count another quantity: the number of pairings of $\{1, 2, 3, 4, 5, 6\}$ that don't pair any consecutive numbers. Each such pairing gives rise to 48 ways of ordering of these three couples while keeping no couple together. To see why, note that given a pairing, we can choose which couples go into the indices of which pair in $3! = 6$ ways. Since no consecutive numbers are paired no couple stays together. We can order each couple among themselves in 2 different ways, and since there are 3 couples, the number of orderings we get from each such pairing is $6 \cdot 2^3 = 48$.

   We already counted the number of pairings of $\{1, 2, 3, 4, 5, 6\}$ in a previous HW, and the answer is 15. Out of these pairings, it's easy to check that only $\{(1, 3), (2, 5), (4, 6)\}, \{(1, 4), (2, 5), (3, 6)\}, \{(1, 4), (2, 6), (3, 5)\}, \{(1, 5), (2, 4), (3, 6)\}$ $\{(1, 6), (2, 4), (3, 5)\}$ satisfy our criteria. Therefore, there are $5 \cdot 48 = 240$ permutations that leave some couple together. Since there are 720 permutations in total, there are $720 - 240 = 480$ permutations that leave some couple together.

   (d) Find the number of arrangements of "CINCINNATI".
   Solution: If every letter were distinct, the answer would be 10!. However, given any arrangement of these letters, the 3 Is, 3 Ns and the 2 Cs can be permuted among themselves without changing the arrangement. There are $3! \cdot 3! \cdot 2!$ ways of doing this, therefore the answer is

$$\frac{10!}{3! \cdot 3! \cdot 2!}$$