Concepts of Math: Recitation 12

October 7, 2015

Bijective Functions and Cardinality

1. Let $f : A \to B$ be a bijection, where A and B are subsets of $\mathbb{R}$. Prove that, if f is increasing on A, then f^{-1} is increasing on B.

2. Count the bijections from A to B, given that $|A| = |B| = n$.

3. Let A and B be two countable sets. Prove that $A \cup B$ and $A \times B$ are countable.

4. Prove that in the xy-plane there is a circle centered at the origin that passes through no points whose coordinates are both rational numbers.

5. Let I denote the closed unit interval $[0, 1]$ and $I \times I$ denote the closed unit square. Prove that $|I| = |I \times I|$.

6. Consider functions $f : A \to B$ and $g : B \to C$. Answer each question below by providing a proof or a counterexample.

   (a) If $f(g(y)) = y$ for all $y \in B$, does it follow that f is a bijection?

   (b) If $g(f(x)) = x$ for all $x \in A$, does it follow that $f(g(y)) = y$ for all $y \in B$?

7. Verify that
$$f(x) = \frac{2x - 1}{2x(1 - x)}$$
defines a bijection from $(0, 1)$ to $\mathbb{R}$. In the proof that f is surjective, use the quadratic formula.

8. Consider $f : A \to A$. Prove that if $f \circ f$ is injective, then f is injective.