18-461/661: Introduction to ML for Engineers

Probability and Linear Algebra Review

Spring 2020

ECE – Carnegie Mellon University

Registration

- Course is currently full and we can't increase class size
- Expect several students will drop the course
- Course will be offered again in Fall 2020

Direct all waitlist-related questions to Megan Oliver (Pittsburgh) or Brittany Walker (SV): mvoliver@andrew.cmu.edu and bmw2@andrew.cmu.edu

- If you're not registered, we encourage you to stay patient
- You are welcome to keep attending the lectures until the waitlists are sorted out

Announcements

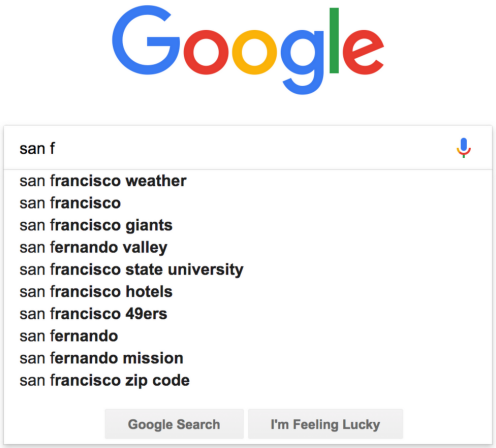
- Graded quizzes will be posted on Gradescope – Entry Code 9NEDVR
- HW1 will be posted on Fri Jan 17, due on Mon Jan 27
- Recitation this Friday will go through the math quiz
- No class on Monday due to the MLK Day holiday

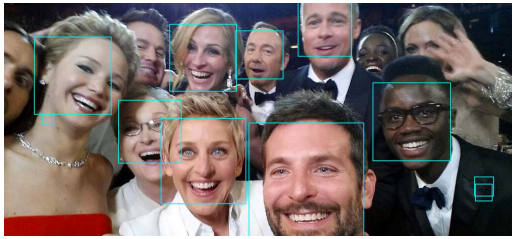
Outline

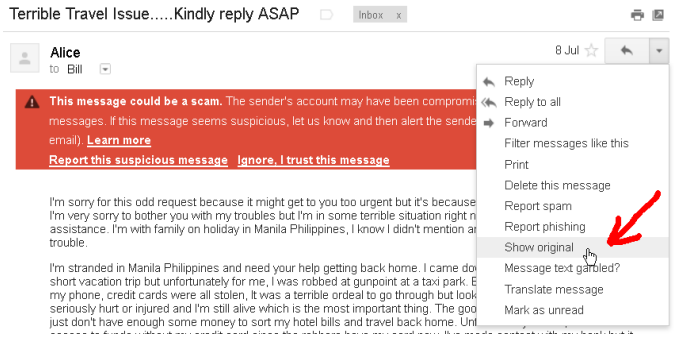
1. Recap: What is Machine Learning?
2. Probability Review
3. A Simple Learning Problem: MLE/MAP Estimation
4. Linear Algebra Review

Recap: What is Machine Learning?

Definition and Examples

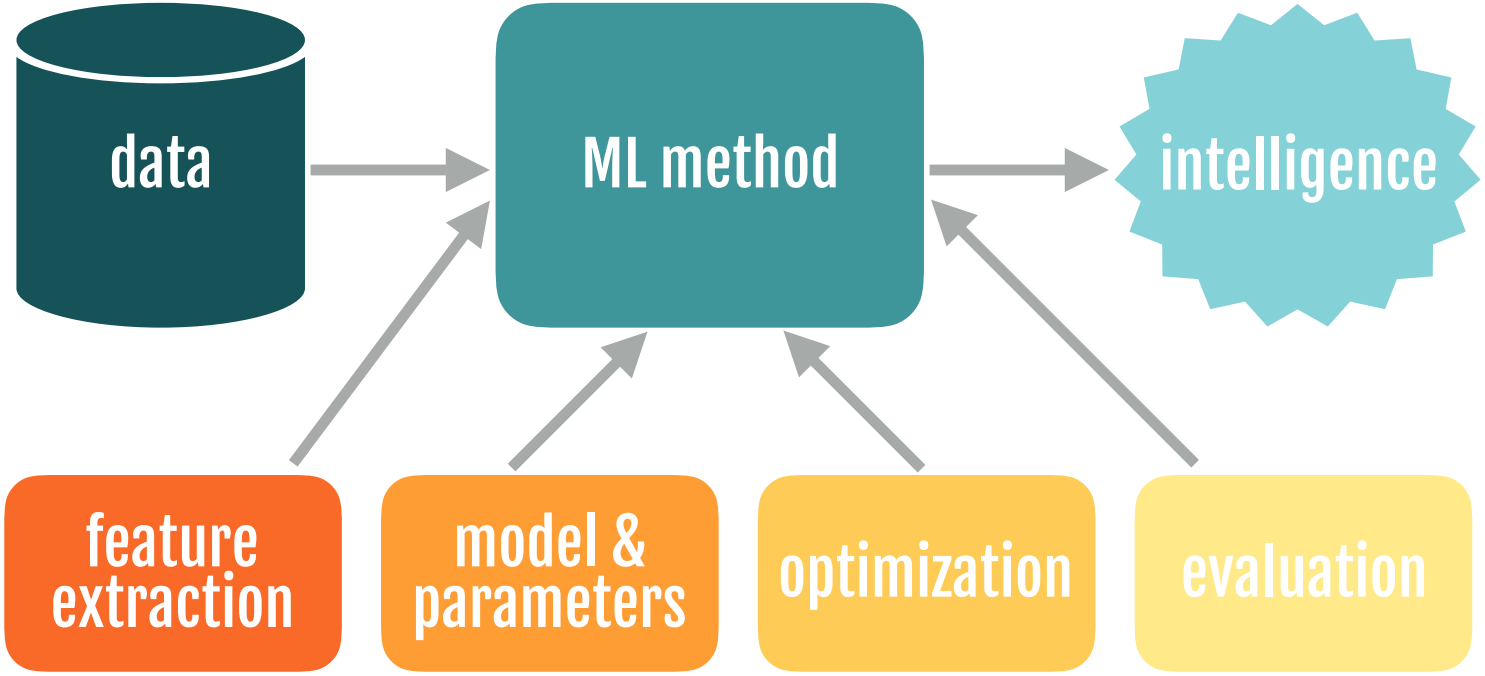






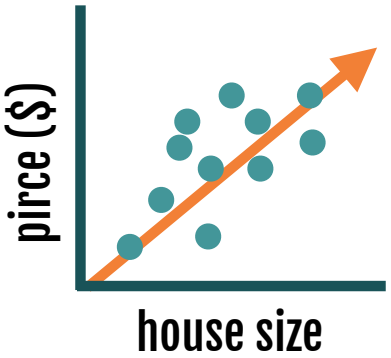
- **Machine learning is**: the study of methods that *improve their performance on some task with experience*

Goal: Choose the Right ML Method for a Given Task



Task 1: Regression

How much should you sell your house for?

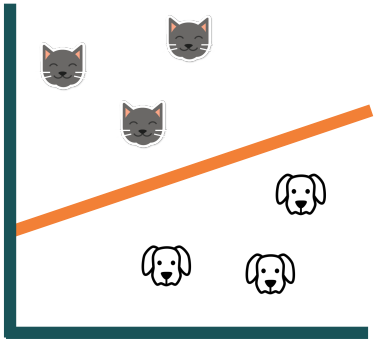


input: houses & features **learn**: $x \to y$ relationship **predict**: y (*continuous*)

Course Covers: Feature Scaling, Linear/Ridge Regression, Loss Function, SGD, Regularization, Cross Validation

Task 2: Classification

Cat or dog?



= ??

input: cats and dogs

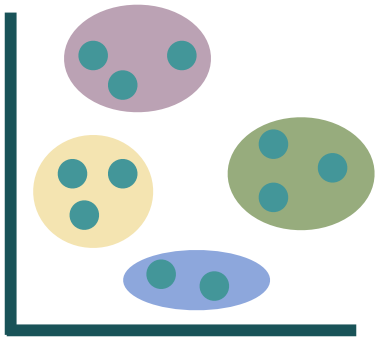
learn: $x \rightarrow y$ relationship

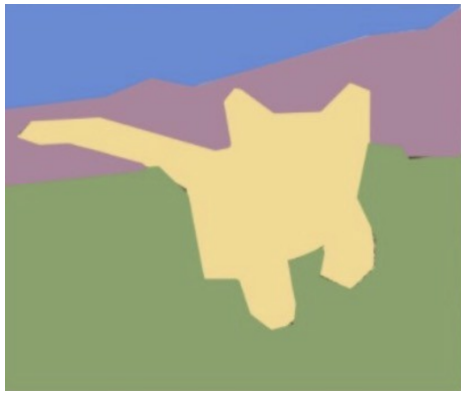
predict: y (*categorical*)

Course Covers: Naive Bayes, Logistic Regression, SVMs, Neural Nets, Decision Trees, Boosting

Task 3: Clustering

How to segment an image?



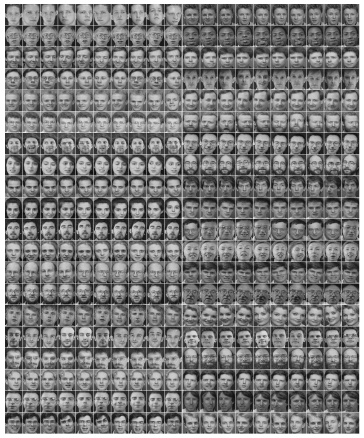


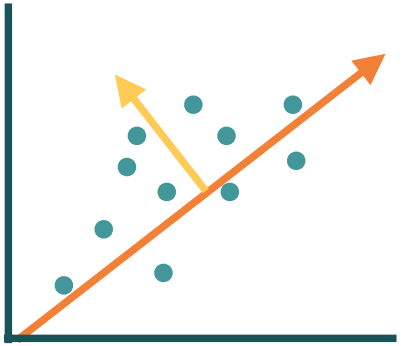
input: raw pixels $\{x\}$ **separate**: $\{x\}$ into sets **output**: cluster labels $\{z\}$

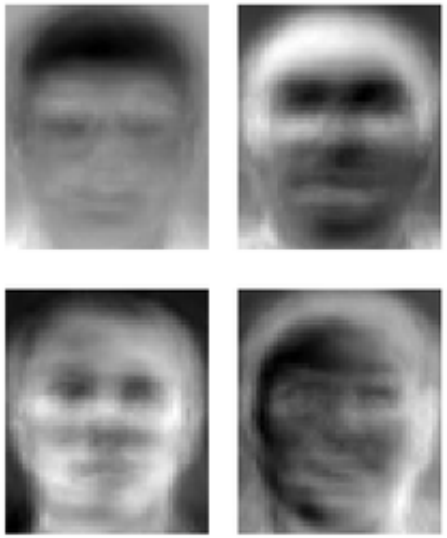
Course Covers: Nearest Neighbors, K-means clustering

Task 4: Embedding

How to reduce size of dataset?







input: large dataset $\{x\}$ **find**: sources of variation **return**: representation $\{z\}$

Course Covers: Dimensionality Reduction, PCA

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Probability Review

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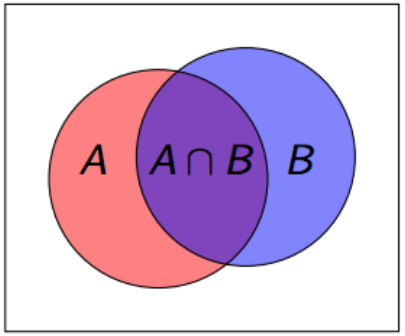
Axioms of Probability



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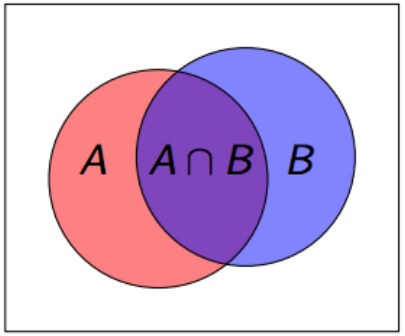

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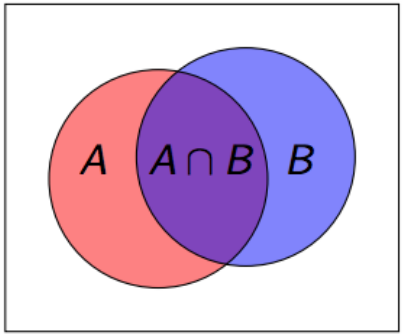

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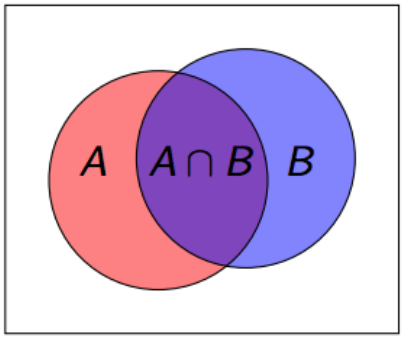
$$P(A \cup B) = 0.75 + 0.5 - 0.5 = 0.75$$

Conditional Probability



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$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

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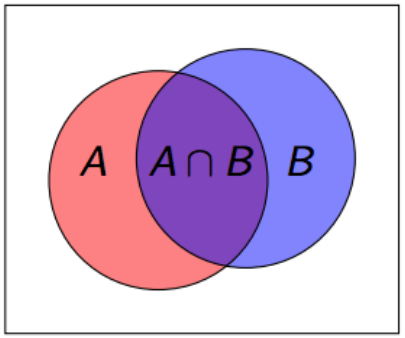

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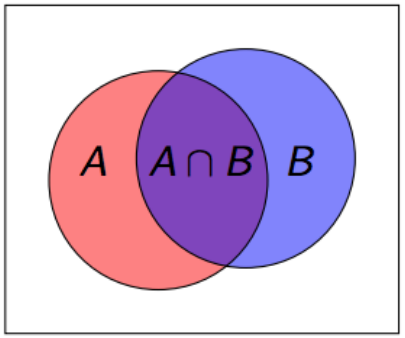

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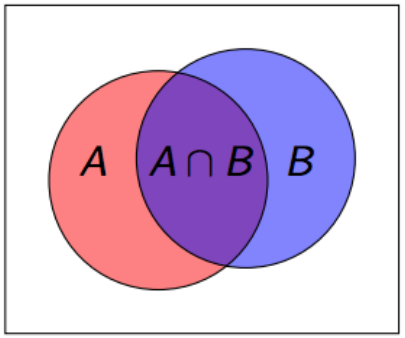

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- Bayes rule:

$$P(B \mid A)P(A) = P(A \cap B) = P(A \mid B)P(B)$$

$$\implies P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Some Other Concepts that You Should Know

- Discrete and Continuous Random Variables
- PMF, PDF, CDF
- Expectation and Variance
- Entropy

A Simple Learning Problem: MLE/MAP Estimation

Dogecoin

- Scenario: You find a coin on the ground.

- *You ask yourself: Is this a fair or biased coin? What is the probability that I will flip a heads?*

- You flip the coin 10 times . . .

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- It comes up as 'H' 8 times and 'T' 2 times

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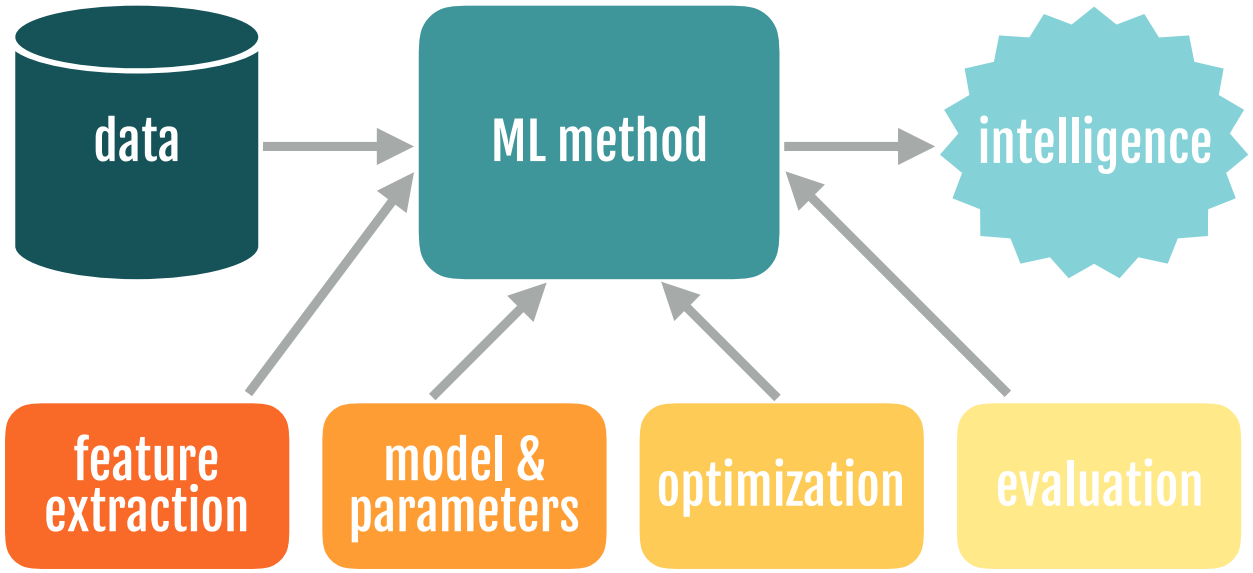
- Can we learn the bias of the coin from this data?

Recall: Machine Learning Pipeline



Two approaches that we will discuss today:

- Maximum likelihood Estimation (MLE)
- Maximum a posteriori Estimation (MAP)

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- Question: Given this model and the data we've observed, can we calculate an estimate of θ?
- **MLE**: Choose θ that maximizes the *likelihood* of the observed data

$$\hat{\theta}_{MLE} = \arg\max_{\theta} P(D \mid \theta)$$

How to solve?

- $\log(x)$ is a monotone increasing function; will not affect the arg max

$$\begin{aligned}\hat{\theta}_{MLE} &= \arg\max_{\theta} P(D \mid \theta) \\ &= \arg\max_{\theta} \log P(D \mid \theta) \\ &= \arg\max_{\theta} \log\left(\theta^{n_H}(1-\theta)^{n_T}\right)\end{aligned}$$

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- Take derivative $\frac{\partial}{\partial\theta} \log P(D \mid \theta)$ and set equal to zero

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- You flip the coin 10 times ...

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Here, we are trusting the data completely. But there could be too little data or noisy data

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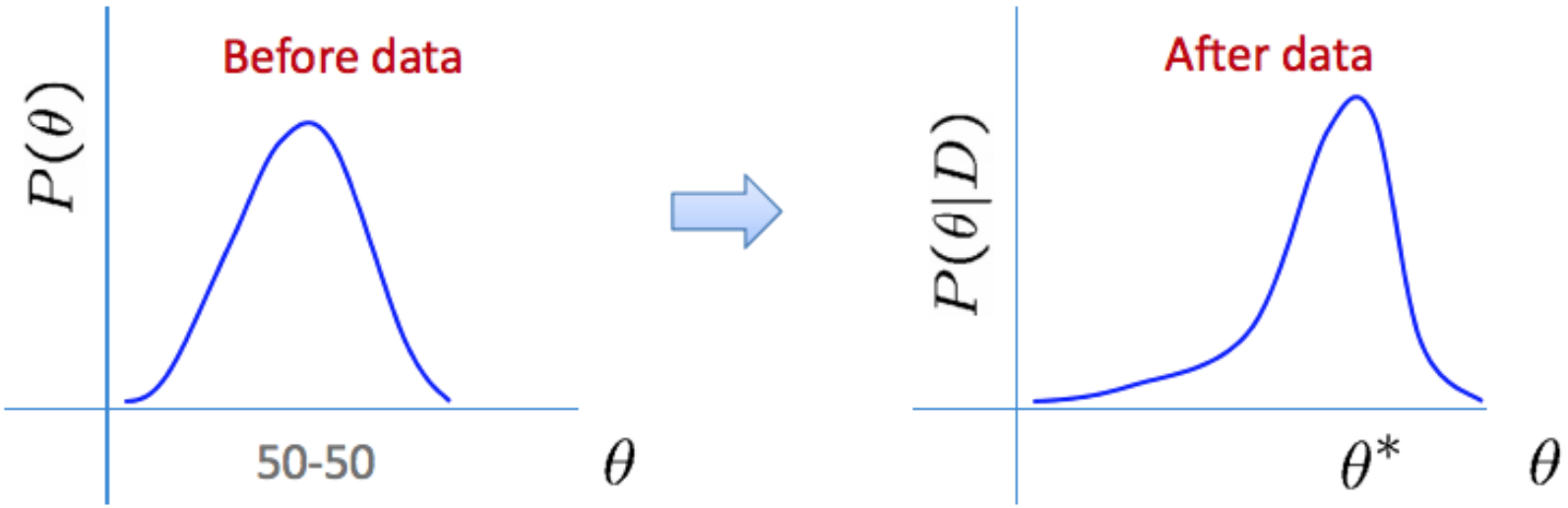

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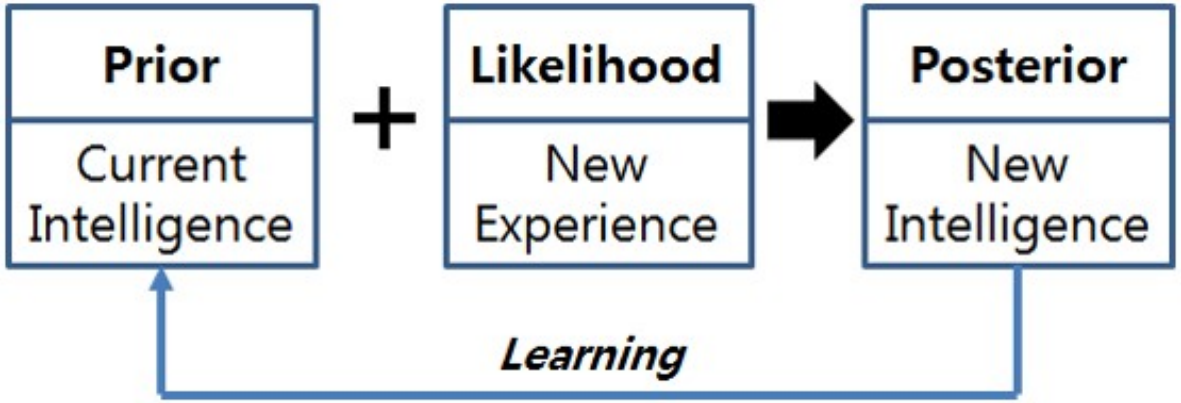

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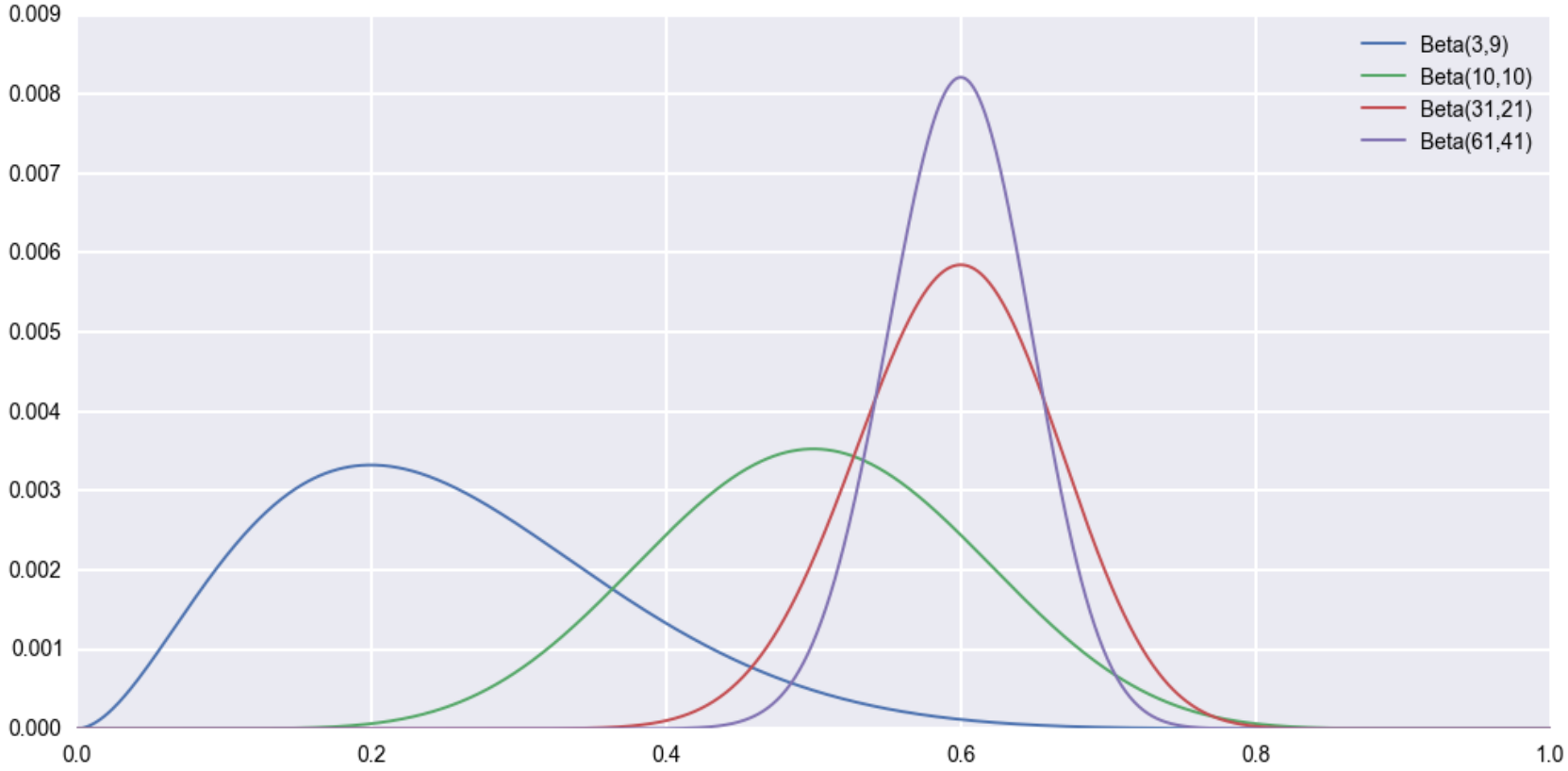
- Interpretation: α = number of expected heads, β = number of expected tails. Larger value of $\alpha + \beta$ denotes more confidence (and smaller variance).

Beta Distribution



$\frac{\alpha}{\beta}$ controls left/right bias, $\alpha + \beta$ controls height of peak

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- Note that as $n_H + n_T \to \infty$, the effect of the prior disappears and we recover the MLE estimate

Putting it all together

$$\hat{\theta}_{MLE} = \frac{n_H}{n_H + n_T}$$

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Bayesians vs. Frequentists

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with experience (the more coin flips we see, the better our guess)

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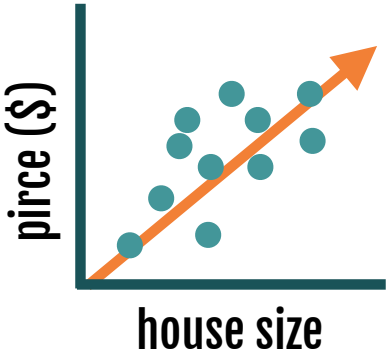
Key idea: these are *choices*. It's important to understand the implications of these choices and evaluate their trade-offs for the problem at hand.

Linear Algebra Review

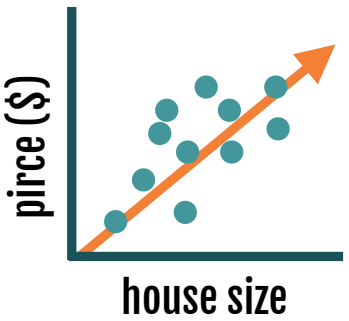
Recall: Task 1: Regression

How much should you sell your house for?



input: houses & features **learn**: $x \to y$ relationship **predict**: y (*continuous*)

Data Can be Compactly Represented by Matrices



- Learn parameters (w_1, w_0) of the orange line $y = w_1 x + w_0$

Sq.ft

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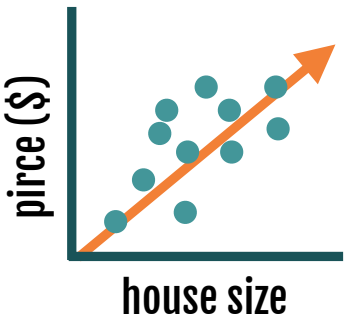

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- Can represent compactly in matrix notation

$$\begin{bmatrix} 1000 & 1 \\ 2000 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} = \begin{bmatrix} 200,000 \\ 350,000 \end{bmatrix}$$

Some Concepts That You Should Know

- Invertibility of Matrices and Computing Inverses
- Vector Norms – L2, Frobenius etc., Inner Products
- Eigenvalues and Eigen-vectors
- Singular Value Decomposition
- Covariance Matrices and Positive Semi-definite-ness

Excellent Resources:

- Essence of Linear Algebra YouTube Series by 3Blue1Brown
- Prof. Gilbert Strang's course at MIT

Matrix multiplication

- For two matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, their product is:

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  - Matrix-vector product: $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n \iff Ax \in \mathbb{R}^m$

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix}, \; Ax \in \mathbb{R}^m = \sum_{i=1}^{n} a_i x_i$$

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Norms and Loss Functions

- You could have data from many houses

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$$A \qquad \times \qquad w = \qquad y$$

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- There isn't a $w = [w_1, w_0]^T$ that will satisfy all equations
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- But since this a vector, we need an operator that can map the vector $y - Aw$ to a scalar

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- Answer: $\|y - Aw\|_1 = 0.5$

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- If the eigen-vectors are linearly independent, we can express A as

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Outline

1. Recap: What is Machine Learning?
2. Probability Review
3. A Simple Learning Problem: MLE/MAP Estimation
4. Linear Algebra Review