# 18-661 Introduction to Machine Learning

## Naive Bayes

Spring 2020

ECE – Carnegie Mellon University

# Announcements

- HW 2 is due in one week (on Monday, February 10).
- Python tutorial has been rescheduled for this week; rooms and times TBD.
- Please refer to the pinned Piazza post on "Resources to refresh math background" (or come to office hours!) if you are struggling with probability, linear algebra, or matrix/vector calculus.
- HW 1 solutions have been posted to Canvas.

# Outline

1. Review of Hyperparameter Tuning and the Bias-Variance Trade-off
2. Classification Example: Spam Detection
3. Naive Bayes Model
4. Parameter Estimation

# Review of Hyperparameter Tuning and the Bias-Variance Trade-off

# Tuning by using a validation dataset

**Training data are used to learn $f(\cdot)$.**

N samples/instances: $\mathcal{D}^{\text{TRAIN}} = \{(\boldsymbol{x}_1, y_1), (\boldsymbol{x}_2, y_2), \cdots, (\boldsymbol{x}_\mathrm{N}, y_\mathrm{N})\}$

**Test data are used to assess the prediction error.**

- M samples/instances: $\mathcal{D}^{\text{TEST}} = \{(\boldsymbol{x}_1, y_1), (\boldsymbol{x}_2, y_2), \cdots, (\boldsymbol{x}_\mathrm{M}, y_\mathrm{M})\}$
- They are used for assessing how well $f(\cdot)$ will do in predicting an unseen $\boldsymbol{x} \notin \mathcal{D}^{\text{TRAIN}}$.

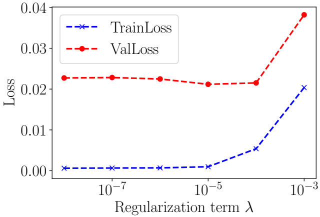
**Validation data are used to optimize hyperparameter(s).**

L samples/instances: $\mathcal{D}^{\text{VAL}} = \{(\boldsymbol{x}_1, y_1), (\boldsymbol{x}_2, y_2), \cdots, (\boldsymbol{x}_\mathrm{L}, y_\mathrm{L})\}$

Training data, validation and test data should not overlap!

# Tuning the Hyperparameter $\lambda$

- For each possible value of the hyperparameter (say $\lambda = 1, 3, \cdots, 100$)
  - Train a model using $\mathcal{D}^{\text{TRAIN}}$
  - Evaluate the performance of the model on $\mathcal{D}^{\text{VAL}}$
- Choose the model with the best performance on $\mathcal{D}^{\text{VAL}}$
- Evaluate the model on $\mathcal{D}^{\text{TEST}}$

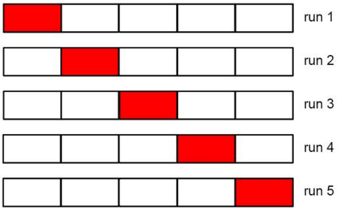


# Cross-validation

**What if we do not have validation data?**

- We split the training data into S equal parts.
- We use each part in turn as a validation dataset and use the others as a training dataset.
- We choose the hyperparameter such that the model performs the best (based on average, variance, etc.).
- Finally, retrain the model on the entire training dataset with this "best" hyperparameter.

**Figure 1:** $S = 5$: 5-fold cross validation




Special case: when $S = N$, this will be leave-one-out.

# Bias-Variance Trade-off: Intuition

- High Bias: Model is not rich enough to fit the training dataset and achieve low training loss
- High Variance: If the training dataset changes slightly, the model changes a lot
- Regularization helps find a middle ground

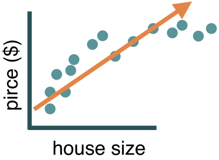



**Figure 2:** High Bias

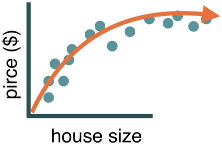



**Figure 3:** Just Right

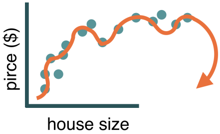



**Figure 4:** High Variance

# Bias/variance tradeoff for regression

**Goal: to understand the sources of prediction errors**

- $\mathcal{D}$: our training data
- $h_{\mathcal{D}}(\boldsymbol{x})$: our prediction function
  We are using the subscript $\mathcal{D}$ to indicate that the prediction function is learned on the specific set of training data $\mathcal{D}$.
- $\ell(h(\boldsymbol{x}), y)$: our square loss function for regression

$$\ell(h_{\mathcal{D}}(\boldsymbol{x}), y) = [h_{\mathcal{D}}(\boldsymbol{x}) - y]^2$$

- Unknown joint distribution $p(\boldsymbol{x}, y)$

# The effect of finite training samples

**Every training dataset $\mathcal{D}$ is a sample from the following joint distribution of all possible training datasets.**

$$\mathcal{D} \sim P(\mathcal{D}) = \prod_{n=1}^{N} p(\boldsymbol{x}_n, y_n)$$

Thus, the prediction function $h_{\mathcal{D}}(\boldsymbol{x})$ is a random function with respect to this distribution. So is also its risk

$$R[h_{\mathcal{D}}(\boldsymbol{x})] = \int_{\boldsymbol{x}} \int_{y} [h_{\mathcal{D}}(\boldsymbol{x}) - y]^2 p(\boldsymbol{x}, y) d\boldsymbol{x} dy$$

# Three components of average risk

**The average risk (with quadratic loss) can be decomposed as:**

$$\mathbb{E}_{\mathcal{D}} R[h_{\mathcal{D}}(\boldsymbol{x})] = \underbrace{\int_{\mathcal{D}} \int_{\boldsymbol{x}} \int_{y} [h_{\mathcal{D}}(\boldsymbol{x}) - \mathbb{E}_{\mathcal{D}} h_{\mathcal{D}}(\boldsymbol{x})]^2 p(\boldsymbol{x}, y) d\boldsymbol{x} dy \; P(\mathcal{D}) d\mathcal{D}}_{\text{VARIANCE: error due to training dataset}}$$

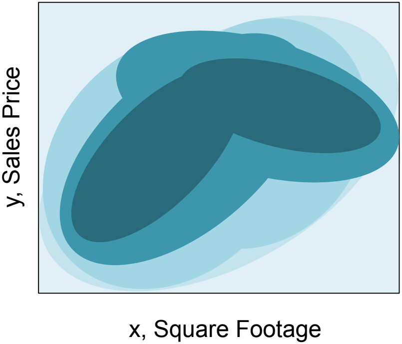
$$+ \underbrace{\int_{\boldsymbol{x}} \int_{y} [\mathbb{E}_{\mathcal{D}} h_{\mathcal{D}}(\boldsymbol{x}) - \mathbb{E}_{y}[y|\boldsymbol{x}]]^2 p(\boldsymbol{x}, y) d\boldsymbol{x} dy}_{\text{BIAS}^2\text{: error due to the model approximation}}$$

$$+ \underbrace{\int_{\boldsymbol{x}} \int_{y} [\mathbb{E}_{y}[y|\boldsymbol{x}] - y]^2 p(\boldsymbol{x}, y) d\boldsymbol{x} dy}_{\text{NOISE: error due to randomness of } y}$$

Here we define: $h_{\mathcal{D}}(\boldsymbol{x})$ as the output of the model trained on $\mathcal{D}$, $\mathbb{E}_{\mathcal{D}} h_{\mathcal{D}}(\boldsymbol{x})$ as the expectation of the model over all datasets $\mathcal{D}$, and $\mathbb{E}_{y}[y|\boldsymbol{x}]$ as the expected value of $y$ conditioned on $\boldsymbol{x}$.

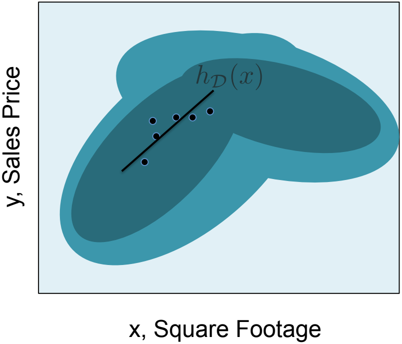
## Bias-Variance Trade-off: Illustration

- Joint distribution of square footage $x$ and house sales price $y$
- Darker color indicates higher probability regions



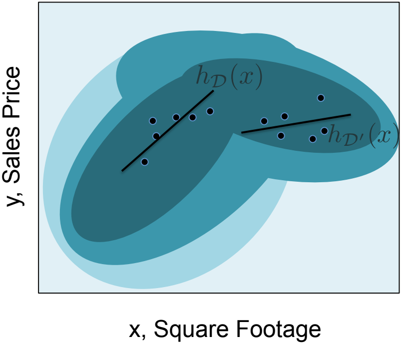
# Bias-Variance Trade-off: Illustration

- We have access to dataset $\mathcal{D}$ sampled from the joint distribution
- Learn linear model $h_{\mathcal{D}}(x)$ based on $\mathcal{D}$



# Bias-Variance Trade-off: Illustration

- Repeating this process for a different dataset $\mathcal{D}'$ yields a new linear model $h_{\mathcal{D}}(x)$
- Average of such models over infinitely many datasets sampled from the joint distribution is $\mathbb{E}_{\mathcal{D}} h_{\mathcal{D}}(\boldsymbol{x})$



# Bias-Variance Trade-off: Illustration

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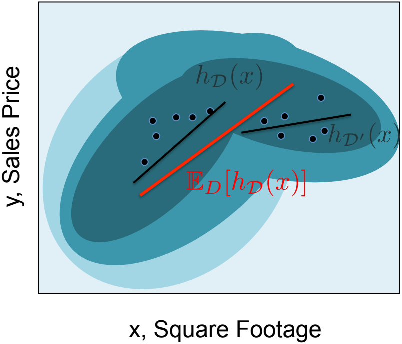
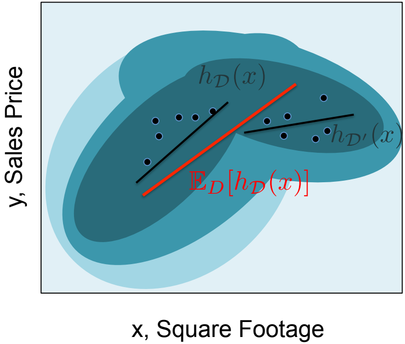


# Bias-Variance Trade-off: Illustration

- Average of such models over infinitely many datasets sampled from the joint distribution is $\mathbb{E}_{\mathcal{D}} h_{\mathcal{D}}(\boldsymbol{x})$
- Variance term captures how much individual models differ from the average $\underbrace{\int_{\mathcal{D}} \int_{\boldsymbol{x}} \int_{y} [h_{\mathcal{D}}(\boldsymbol{x}) - \mathbb{E}_{\mathcal{D}} h_{\mathcal{D}}(\boldsymbol{x})]^2 p(\boldsymbol{x}, y) d\boldsymbol{x} dy \; P(\mathcal{D}) d\mathcal{D}}_{\text{VARIANCE: error due to training dataset}}$

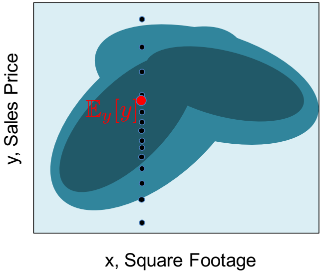


# Bias-Variance Trade-off: Illustration

- For a given $\boldsymbol{x}$, we have a conditional distribution $p(y|\boldsymbol{x})$ of sales prices; the Bayesian optimal prediction of the label value for a given $\boldsymbol{x}$ is $\mathbb{E}_y[y|\boldsymbol{x}]$;
- The noise term measures the inherent variance in labels $y$

$$\underbrace{\int_{\boldsymbol{x}}\int_{y}[\mathbb{E}_y[y|\boldsymbol{x}] - y]^2 p(\boldsymbol{x}, y) d\boldsymbol{x} dy}_{\text{NOISE: error due to randomness of } y}$$

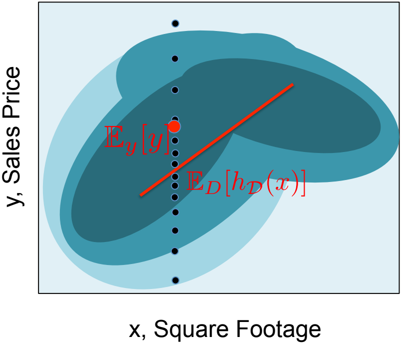
- This term has nothing to do with our prediction $h_{\mathcal{D}}(\boldsymbol{x})$



# Bias-Variance Trade-off: Illustration

- If our model class was rich enough (eg. a high-degree polynomial), then $\mathbb{E}_{\mathcal{D}} h_{\mathcal{D}}(\boldsymbol{x})$ should perfectly match $\mathbb{E}_y[y|\boldsymbol{x}]$
- Restricting to simpler models (eg. linear) results in a bias

$$\underbrace{\int_{\boldsymbol{x}}\int_y [\mathbb{E}_{\mathcal{D}} h_{\mathcal{D}}(\boldsymbol{x}) - \mathbb{E}_y[y|\boldsymbol{x}]]^2 p(\boldsymbol{x}, y) d\boldsymbol{x} dy}_{\text{BIAS}^2\text{: error due to the model approximation}}$$



# Outline

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2. Classification Example: Spam Detection
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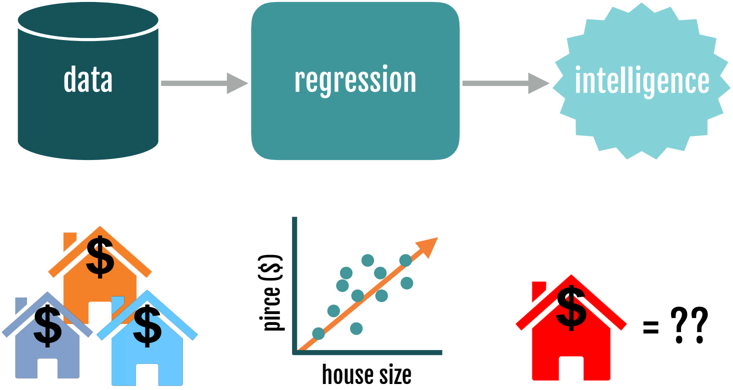
# Classification Example: Spam Detection

# Task 1: Regression (so far we studied this)

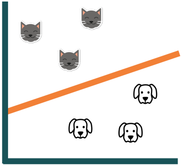
*How much should you sell your house for?*

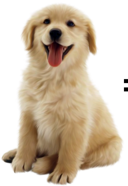



**input**: houses & features **learn**: $x \to y$ relationship **predict**: $y$ (*continuous*)

# Task 2: Classification (next topic)

*Cat or dog?*



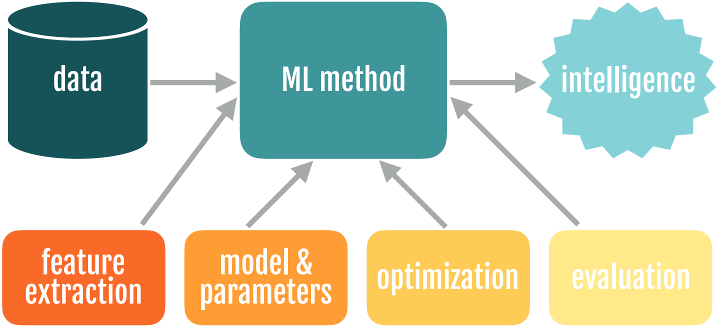
= ??

**input**: cats and dogs

**learn**: $x \rightarrow y$ relationship

**predict**: $y$ (*categorical*)

# Goal: Learn about ML pipeline



# I'm going to be rich!!

FROM THE DESK OF MR. AMINU SALEH
DIRECTOR, FOREIGN OPERATIONS DEPARTMENT
AFRI BANK PLC
Afribank Plaza,
14th Floormoney344.jpg
51/55 Broad Street,
P.M.B 12021 Lagos-Nigeria


Attention: Honorable Beneficiary,

IMMEDIATE PAYMENT NOTIFICATION VALUED AT **US$10 MILLION**

It is my modest obligation to write you this letter in regards to the authorization of your owed payment through our most respected financial institution (AFRI BANK PLC). I am Mr. Aminu Saleh, The Director, Foreign Operations Department, AFRI Bank Plc, NIGERIA. The British Government, in conjunction with the US GOVERNMENT, WORLD BANK, UNITED NATIONS ORGANIZATION on foreign payment matters, has empowered my bank after much consultation and consideration, to handle all foreign payments and release them to their appropriate beneficiaries with the help of a representative from Federal Reserve Bank.

To facilitate the process of this transaction, please kindly re-confirm the following information below:

1) Your full Name and Address:
2) Phones, Fax and Mobile No. :
3) Profession, Age and Marital Status:
4) Copy of any valid form of your Identification:

# How to tell spam from ham?

FROM THE DESK OF MR. AMINU SALEH
DIRECTOR, FOREIGN OPERATIONS DEPARTMENT
AFRI BANK PLC
Afribank Plaza,
14th Floormoney344.jpg
51/55 Broad Street,
P.M.B 12021 Lagos-Nigeria

Attention: Honorable Beneficiary,

IMMEDIATE PAYMENT NOTIFICATION VALUED AT **US$10 MILLION**


Hi Virginia,

Can we meet today at 2pm?

thanks,

Carlee

# How might we create features?

Intuition

- Q: How might a human solve this problem?
- A: Simple strategy would be to look for keywords that we often associate with spam

Spam

- We expect to see words like "money", "free", "bank account"

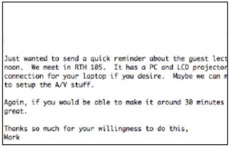
Ham

- Expect to see less spam words, more personalization (e.g., name)

# Simple strategy: Count the words

**Bag-of-word representation of documents (and textual data)**

$$\begin{pmatrix} \text{free} & 100 \\ \text{money} & 2 \\ \vdots & \vdots \\ \text{account} & 2 \\ \vdots & \vdots \end{pmatrix}$$

Just wanted to send a quick reminder about the guest lect
noon. We meet in RTH 105. It has a PC and LCD projector
connection for your laptop if you desire. Maybe we can m
to setup the A/V stuff.

Again, if you would be able to make it around 30 minutes
great.

Thanks so much for your willingness to do this,
Mark

$$\begin{pmatrix} \text{free} & 1 \\ \text{money} & 1 \\ \vdots & \vdots \\ \text{account} & 2 \\ \vdots & \vdots \end{pmatrix}$$

# Weighted sum of those telltale words

$$\begin{pmatrix} 100 \times 0.2 \\ 2 \times 0.3 \\ \vdots \\ 2 \times 0.3 \\ \vdots \end{pmatrix}$$

different weights for spam and ham: representing how compatible the word pattern is to each category

$$\begin{pmatrix} \text{free} & 100 \\ \text{money} & 2 \\ \vdots & \vdots \\ \text{account} & 2 \\ \vdots & \vdots \end{pmatrix}$$

$$\begin{pmatrix} 100 \times 0.01 \\ 2 \times 0.02 \\ \vdots \\ 2 \times 0.01 \\ \vdots \end{pmatrix}$$

# Weighted sum of those telltale words

$$\begin{pmatrix} \text{free} & 100 \\ \text{money} & 2 \\ \vdots & \vdots \\ \text{account} & 2 \\ \vdots & \vdots \end{pmatrix}$$

$$\begin{pmatrix} 100 \times 0.2 \\ 2 \times 0.3 \\ \vdots \\ 2 \times 0.3 \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} 100 \times 0.01 \\ 2 \times 0.02 \\ \vdots \\ 2 \times 0.01 \\ \vdots \end{pmatrix}$$

different weights for spam and ham: representing how compatible the word pattern is to each category

= 3.2


= 1.03

# Our intuitive model of classification

1. **Assign weight to each word**
   - Let $s$:= spam weights, $h$:= ham weights
   - Compute compatibility score of **spam**
     - $(\#$ "free" $\times s_{\text{free}}) + (\#$ "account" $\times s_{\text{account}}) + (\#$ "money" $\times s_{\text{money}})$
   - Compute compatibility score of **ham**
     - $(\#$ "free" $\times h_{\text{free}}) + (\#$ "account" $\times h_{\text{account}}) + (\#$ "money" $\times h_{\text{money}})$
2. **Make a decision**
   - if spam score $>$ ham score then spam
   - else ham

# Your turn

Suppose you see the following email:

```
CONGRATULATIONS!! Your email address have won you the
lottery sum of US$2,500,000.00 USD to claim your prize,
contact your office agent (Athur walter) via email
claims2155@yahoo.com.hk or call +44 704 575 1113
```

And our weights for spam and ham are:
spam: [lottery=0.3, prize=0.3, office=0.01, email=0.01, ...]
ham: [lottery=0.01, prize=0.01, office=0.1, email=0.05, ...]

Will we predict that the email is spam or ham?

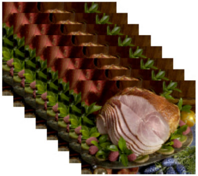
spam = 0.3*1 + 0.3*1 + 0.01*1 + 0.01*2 =0.63
ham = 0.01*1 + 0.01*1 + 0.1*1 + 0.05*2 = 0.22
so we predict spam!

# How do we get the weights?

Learn from experience

- get a lot of spams
- get a lot of hams

But what to optimize?

# Naive Bayes Model

# Naive Bayes model for identifying spam

- **Class label**: binary
  - $y = \{$ spam, ham $\}$
- **Features**: word counts in the document (bag-of-words)
  - $\mathbf{x} = \{(\text{'free'}, 100), (\text{'lottery'}, 5), (\text{'money'}, 10)\}$
  - Each pair is in the format of $(\text{word}_i, \#\text{word}_i)$, namely, a unique word in the dictionary, and the number of times it shows up
- **Model**

$$p(\mathbf{x}|spam) = p(\text{`}free'|spam)^{100} p(\text{`}lottery'|spam)^{5} p(\text{`}money'|spam)^{10} \cdots$$

  - Choose the "most likely" option: $p(\mathbf{x}|spam)p(spam)$ vs. $p(\mathbf{x}|ham)p(ham)$

These conditional probabilities are the parameters we need to estimate

# What is naive about this?

- Strong assumption of conditional independence:

$$p(\text{word}_i, \text{word}_j | y) = p(\text{word}_i | y) p(\text{word}_j | y)$$

- Previous example:

$$p(\mathbf{x}|spam) = p(\text{`}free\text{'}|spam)^{100} p(\text{`}lottery\text{'}|spam)^{5} p(\text{`}money\text{'}|spam)^{10} \cdots$$

- Independence across different words as well as multiple occurrences of the same word
- This assumption makes estimation much easier (as we'll see)

# Naive Bayes classification rule

For any document $\mathbf{x}$, we want to compare $p(\text{spam}|\mathbf{x})$ and $p(\text{ham}|\mathbf{x})$

Recall that by Bayes rule we have:

$$p(\text{spam}|\mathbf{x}) = \frac{p(\mathbf{x}|\text{spam})p(\text{spam})}{p(\mathbf{x})}$$

$$p(\text{ham}|\mathbf{x}) = \frac{p(\mathbf{x}|\text{ham})p(\text{ham})}{p(\mathbf{x})}$$

Denominators are the same, and easier to compute logarithms, so we compare the spam and ham probabilities:

$$\log[p(\mathbf{x}|\text{spam})p(\text{spam})] \quad \text{versus} \quad \log[p(\mathbf{x}|\text{ham})p(\text{ham})]$$

# Classifier in linear form

$$\log[p(\mathbf{x}|\text{spam})p(\text{spam})] = \log\left[\prod_i p(\text{word}_i|\text{spam})^{\#\text{word}_i} p(\text{spam})\right]$$
$$= \sum_i (\#\text{word}_i) \log p(\text{word}_i|\text{spam}) + \log p(\text{spam})$$

Similarly, we have

$$\log[p(\mathbf{x}|\text{ham})p(\text{ham})] = \sum_i (\#\text{word}_i) \log p(\text{word}_i|\text{ham}) + \log p(\text{ham})$$

We're back to the idea of comparing weighted sums of word occurrences!

$\log p(spam)$ *and* $\log p(ham)$ *are called "priors" (in our initial example we did not include them but they are important!)*

# Mini-summary

What we have shown

By assuming a probabilistic model (i.e., Naive Bayes), we are able to derive a decision rule that is consistent with our intuition

Our next step is to learn the parameters from data

What are the parameters to learn?

# Parameter Estimation

# Formal definition of Naive Bayes

General case

Given a random vector $\mathbf{X} \in \mathbb{R}^{\mathsf{K}}$ and a dependent variable $Y \in [\mathsf{C}]$, the Naive Bayes model defines the joint distribution

$$\begin{aligned} P(\mathbf{X} = \mathbf{x}, Y = c) &= P(Y = c) P(\mathbf{X} = \mathbf{x} | Y = c) \\ &= P(Y = c) \prod_{k=1}^{\mathsf{K}} P(\text{word}_k | Y = c)^{x_k} \\ &= \pi_c \prod_{k=1}^{\mathsf{K}} \theta_{ck}^{x_k} \end{aligned}$$

where $\pi_c = P(Y = c)$ is the prior probability of class $c$, $x_k$ is the number of occurences of the $k$th word, and $\theta_{ck} = P(\text{word}_k | Y = c)$ is the weight of the $k$th word for the $c$th class.

# Learning problem

## Training data

$$\mathcal{D} = \{(\mathbf{x}_n, y_n)\}_{n=1}^{\mathsf{N}} \rightarrow \mathcal{D} = \{(\{x_{nk}\}_{k=1}^{\mathsf{K}}, y_n)\}_{n=1}^{\mathsf{N}}$$

## Goal

Learn $\pi_c, c = 1, 2, \cdots, \mathsf{C}$, and $\theta_{ck}, \forall c \in [\mathsf{C}], k \in [\mathsf{K}]$ under the constraints:

$$\sum_c \pi_c = 1$$

and

$$\sum_k \theta_{ck} = \sum_k P(\text{word}_k | Y = c) = 1$$

as well as: all $\pi_c$, $\theta_{ck} \geq 0$.

# Our hammer: Maximum likelihood estimation

Likelihood of the training data

$$\mathcal{D} = \{(\mathbf{x}_n, y_n)\}_{n=1}^{\mathsf{N}} \to \mathcal{D} = \{(\{x_{nk}\}_{k=1}^{\mathsf{K}}, y_n)\}_{n=1}^{\mathsf{N}}$$

$$L = P(\mathcal{D}) = \prod_{n=1}^{\mathsf{N}} \pi_{y_n} P(\mathbf{x}_n | y_n)$$

Log-Likelihood of the training data

$$\mathcal{L} = \log P(\mathcal{D}) = \log \prod_{n=1}^{\mathsf{N}} \pi_{y_n} P(\mathbf{x}_n | y_n)$$

# Our hammer: Maximum likelihood estimation

Log-Likelihood of the training data

$$
\begin{aligned}
\mathcal{L} = \log P(\mathcal{D}) &= \log \prod_{n=1}^{\mathrm{N}} \pi_{y_n} P(\mathbf{x}_n | y_n) \\
&= \log \prod_{n=1}^{\mathrm{N}} \left( \pi_{y_n} \prod_k \theta_{y_n k}^{x_{nk}} \right) \\
&= \sum_n \left( \log \pi_{y_n} + \sum_k x_{nk} \log \theta_{y_n k} \right) \\
&= \sum_n \log \pi_{y_n} + \sum_{n,k} x_{nk} \log \theta_{y_n k}
\end{aligned}
$$

Optimize it!

$$
(\pi_c^*, \theta_{ck}^*) = \arg\max \left( \sum_n \log \pi_{y_n} + \sum_{n,k} x_{nk} \log \theta_{y_n k} \right)
$$

# Separating the optimization variables

Note the separation of parameters in the likelihood

$$\sum_n \log \pi_{y_n} + \sum_{n,k} x_{nk} \log \theta_{y_n k}$$

this implies that $\{\pi_c\}$ and $\{\theta_{ck}\}$ can be estimated separately

Reorganize terms

$$\sum_n \log \pi_{y_n} = \sum_c \log \pi_c \times (\#\text{of data points labeled as c})$$

and

$$\sum_{n,k} x_{nk} \log \theta_{y_n k} = \sum_c \sum_{n:y_n=c} \sum_k x_{nk} \log \theta_{ck} = \sum_c \sum_{n:y_n=c,k} x_{nk} \log \theta_{ck}$$

The latter implies $\{\theta_{ck}\}$ and $\{\theta_{c'k}\}$ for $c \neq c'$ can be estimated independently!

# Estimating $\{\pi_c\}$

We want to maximize

$$\sum_c \log \pi_c \times (\#\text{of data points labeled as c})$$

Intuition

- Similar to roll a dice (or flip a coin): each side of the dice shows up with a probability of $\pi_c$ (total C sides)
- And we have total N trials of rolling this dice

Solution

$$\pi_c^* = \frac{\#\text{of data points labeled as c}}{\text{N}}$$

# Estimating $\{\theta_{ck}, k = 1, 2, \cdots, \mathsf{K}\}$

We want to maximize

$$\sum_{n:y_n=c,k} x_{nk} \log \theta_{ck}$$

Intuition

- Again similar to roll a dice: each side of the dice shows up with a probability of $\theta_{ck}$ (total K sides)
- And we have total $\sum_{n:y_n=c,k} x_{nk}$ trials (times a word shows up in class $c$).

Solution

$$\theta^*_{ck} = \frac{\#\text{of times word } k \text{ shows up in data points labeled as } c}{\#\text{total trials for data points labeled as } c}$$

# Translating back to our problem of detecting spam emails

- Collect a lot of ham and spam emails as training examples
- Estimate the "prior"

$$p(\text{ham}) = \frac{\#\text{of ham emails}}{\#\text{of emails}}, \quad p(\text{spam}) = \frac{\#\text{of spam emails}}{\#\text{of emails}}$$

- Estimate the weights, e.g., $p(\text{funny\_word}|\text{ham})$

$$p(\text{funny\_word}|\text{ham}) = \frac{\#\text{of funny\_word in ham emails}}{\#\text{of words in ham emails}}$$
$$p(\text{funny\_word}|\text{spam}) = \frac{\#\text{of funny\_word in spam emails}}{\#\text{of words in spam emails}}$$

# Example: Spam Classification

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 1 | 2 | 3 | Ham |
| Email 4 | 1 | 2 | 3 | 2 | Ham |

Find ML estimates of parameters $\pi_c$ and $\theta_{ck}$

$$\theta_{spam,free} = Pr(free|spam)$$
$$\theta_{spam,bank} = Pr(bank|spam)$$
$$\theta_{spam,meet} = Pr(meet|spam)$$
$$\theta_{spam,time} = Pr(time|spam)$$
$$\pi_{spam} = Pr(spam)$$

# Example: Spam Classification

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 1 | 2 | 3 | Ham |
| Email 4 | 1 | 2 | 3 | 2 | Ham |

Find ML estimates of parameters $\pi_c$ and $\theta_{ck}$

$$\begin{aligned}
\theta_{spam,free} &= Pr(free|spam) = (5+4)/(5+3+1+1+4+2+1+1) = 9/18 \\
\theta_{spam,bank} &= Pr(bank|spam) = (3+2)/18 = 5/18 \\
\theta_{spam,meet} &= Pr(meet|spam) = 2/18 \\
\theta_{spam,time} &= Pr(time|spam) = 2/18 \\
\pi_{spam} &= Pr(spam) = 2/4
\end{aligned}$$

# Example: Spam Classification

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 1 | 2 | 3 | Ham |
| Email 4 | 1 | 2 | 3 | 2 | Ham |

Find ML estimates of parameters $\pi_c$ and $\theta_{ck}$

$$\theta_{ham,free} = Pr(free|ham)$$
$$\theta_{ham,bank} = Pr(bank|ham)$$
$$\theta_{ham,meet} = Pr(meet|ham)$$
$$\theta_{ham,time} = Pr(time|ham)$$
$$\pi_{ham} = Pr(ham)$$

# Example: Spam Classification

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 1 | 2 | 3 | Ham |
| Email 4 | 1 | 2 | 3 | 2 | Ham |

Find ML estimates of parameters $\pi_c$ and $\theta_{ck}$

$$\theta_{ham,free} = Pr(free|ham) = 3/16$$

$$\theta_{ham,bank} = Pr(bank|ham) = 3/16$$

$$\theta_{ham,meet} = Pr(meet|ham) = 5/16$$

$$\theta_{ham,time} = Pr(time|ham) = 5/16$$

$$\pi_{ham} = Pr(ham) = 2/4$$

# Classification rule

Given an unlabeled point $\mathbf{x} = \{x_k, k = 1, 2, \cdots, \mathrm{K}\}$, how to label it?

$$
\begin{aligned}
y^* &= \arg\max_{c \in [\mathrm{C}]} P(y = c|\mathbf{x}) \\
&= \arg\max_{c \in [\mathrm{C}]} P(y = c)P(\mathbf{x}|y = c) \\
&= \arg\max_c [\log \pi_c + \sum_k x_k \log \theta_{ck}]
\end{aligned}
$$

Choose class c that maximizes the log-likelihood of an observed email

# Example: Spam Classification

$$\theta_{spam,free} = Pr(free|spam) = 9/18 \qquad \theta_{ham,free} = Pr(free|ham) = 3/16$$
$$\theta_{spam,bank} = Pr(bank|spam) = 5/18 \qquad \theta_{ham,bank} = Pr(bank|ham) = 3/16$$
$$\theta_{spam,meet} = Pr(meet|spam) = 2/18 \qquad \theta_{ham,meet} = Pr(meet|ham) = 5/16$$
$$\theta_{spam,time} = Pr(time|spam) = 2/18 \qquad \theta_{ham,time} = Pr(time|ham) = 5/16$$
$$\pi_{spam} = Pr(spam) = 2/4 \qquad \pi_{ham} = Pr(ham) = 2/4$$

We observe a new email with the word counts (free, bank, meet, time) = (1,3,4,2). Should it be classified as spam or ham?

$$\begin{aligned}
\log \Pr(spam|\mathbf{x}) &\propto \log\left(\Pr(spam)\cdot \Pr(\mathbf{x}|spam)\right) \\
&= \log\left(\frac{2}{4}\cdot\left(\frac{9}{18}\right)\left(\frac{5}{18}\right)^3\left(\frac{2}{18}\right)^4\left(\frac{2}{18}\right)^2\right) \\
&= -7.99
\end{aligned}$$

# Example: Spam Classification

$$\begin{aligned}
\theta_{spam,free} &= Pr(free|spam) = 9/18 & \theta_{ham,free} &= Pr(free|ham) = 3/16 \\
\theta_{spam,bank} &= Pr(bank|spam) = 5/18 & \theta_{ham,bank} &= Pr(bank|ham) = 3/16 \\
\theta_{spam,meet} &= Pr(meet|spam) = 2/18 & \theta_{ham,meet} &= Pr(meet|ham) = 5/16 \\
\theta_{spam,time} &= Pr(time|spam) = 2/18 & \theta_{ham,time} &= Pr(time|ham) = 5/16 \\
\pi_{spam} &= Pr(spam) = 2/4 & \pi_{ham} &= Pr(ham) = 2/4
\end{aligned}$$

We observe a new email with the word counts (free, bank, meet, time) = (1,3,4,2). Should it be classified as spam or ham?

$$\begin{aligned}
\log \Pr(ham|\mathbf{x}) &\propto \log \Pr(ham) \cdot \Pr(\mathbf{x}|ham) \\
&= \log \left( \frac{2}{4} \cdot \left(\frac{3}{16}\right) \left(\frac{3}{16}\right)^3 \left(\frac{5}{16}\right)^4 \left(\frac{5}{16}\right)^2 \right) \\
&= -6.2399
\end{aligned}$$

# Example: Spam Classification

We observe a new email with the word counts (free, bank, meet, time) = (1,3,4,2). Should it be classified as spam or ham?

$$
\begin{aligned}
\log \Pr(\mathit{spam}|\mathbf{x}) &\propto \log \Pr(\mathit{spam}) \cdot \Pr(\mathbf{x}|\mathit{spam}) \\
&= \log\left(\frac{2}{4} \cdot \left(\frac{9}{18}\right)\left(\frac{5}{18}\right)^3\left(\frac{2}{18}\right)^4\left(\frac{2}{18}\right)^2\right) \\
&= -7.99
\end{aligned}
$$

$$
\begin{aligned}
\log \Pr(\mathit{ham}|\mathbf{x}) &\propto \log \Pr(\mathit{ham}) \cdot \Pr(\mathbf{x}|\mathit{ham}) \\
&= \log\left(\frac{2}{4} \cdot \left(\frac{3}{16}\right)\left(\frac{3}{16}\right)^3\left(\frac{5}{16}\right)^4\left(\frac{5}{16}\right)^2\right) \\
&= -6.2399
\end{aligned}
$$

ANSWER: Ham

# Missing features: Some words never occur in ham emails

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 0 | 2 | 3 | Ham |
| Email 4 | 1 | 0 | 3 | 2 | Ham |

Find ML estimates of parameters $\pi_c$ and $\theta_{ck}$

In this training phase, we can just use all available values and ignore missing values

$$\theta_{spam,free} = Pr(free|spam) = 9/18 \qquad \theta_{ham,free} = Pr(free|ham) = 3/13$$

$$\theta_{spam,bank} = Pr(bank|spam) = 5/18 \qquad \theta_{ham,bank} = Pr(bank|ham) = 0/13$$

$$\theta_{spam,meet} = Pr(meet|spam) = 2/18 \qquad \theta_{ham,meet} = Pr(meet|ham) = 5/13$$

$$\theta_{spam,time} = Pr(time|spam) = 2/18 \qquad \theta_{ham,time} = Pr(time|ham) = 5/13$$

$$\pi_{spam} = Pr(spam) = 2/4 \qquad \pi_{ham} = Pr(ham) = 2/4$$

# Missing features: Test phase

$$\theta_{spam,free} = Pr(free|spam) = 9/18 \qquad \theta_{ham,free} = Pr(free|ham) = 3/13$$
$$\theta_{spam,bank} = Pr(bank|spam) = 5/18 \qquad \theta_{ham,bank} = Pr(bank|ham) = 0/13$$
$$\theta_{spam,meet} = Pr(meet|spam) = 2/18 \qquad \theta_{ham,meet} = Pr(meet|ham) = 5/13$$
$$\theta_{spam,time} = Pr(time|spam) = 2/18 \qquad \theta_{ham,time} = Pr(time|ham) = 5/13$$
$$\pi_{spam} = Pr(spam) = 2/4 \qquad \pi_{ham} = Pr(ham) = 2/4$$

New email with the word counts (free, bank, meet, time) = (1,3,4,2),

$$\begin{aligned}
\log \Pr(ham|\mathbf{x}) &\propto \log \Pr(ham) \cdot \Pr(\mathbf{x}|ham) \\
&= \log \left( \frac{2}{4} \cdot \left(\frac{3}{13}\right) \left(\frac{0}{13}\right)^3 \left(\frac{5}{13}\right)^4 \left(\frac{5}{13}\right)^2 \right) \\
&= -\infty
\end{aligned}$$

Problem: The email is ALWAYS classified as spam. Just because an event has not happened yet, doesn't mean that it won't ever happen.

# Dealing with missing features: Solution 1

Remove the features that take zero values for one or more classes

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 0 | 2 | 3 | Ham |
| Email 4 | 1 | 0 | 3 | 2 | Ham |

Remove the 'bank' column

# Dealing with missing features: Solution 1

Remove the features that take zero values for one or more classes

| | free | meet | time | y |
|---|---|---|---|---|
| Email 1 | 5 | 1 | 1 | Spam |
| Email 2 | 4 | 1 | 1 | Spam |
| Email 3 | 2 | 2 | 3 | Ham |
| Email 4 | 1 | 3 | 2 | Ham |

We can then use the same procedure as before to learn the parameters, and then use these parameters to classify new emails

But then we are wasting a lot of useful data..

# Dealing with missing features: Solution 2

Use Laplacian smoothing: Pretend you've seen each word 1 extra time for each class. This is called a 'pseudo-count'.

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 0 | 2 | 3 | Ham |
| Email 4 | 1 | 0 | 3 | 2 | Ham |

$$\theta_{ham,free} = Pr(free|ham) = 3/13$$

$$\theta_{ham,bank} = Pr(bank|ham) = 0/13$$

$$\theta_{ham,meet} = Pr(meet|ham) = 5/13$$

$$\theta_{ham,time} = Pr(time|ham) = 5/13$$

$$\pi_{ham} = Pr(ham) = 2/4$$

# Dealing with missing features: Solution 2

Use Laplacian smoothing: Pretend you've seen each word 1 extra time for each class (spam and ham). This is called a 'pseudo-count'.

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 0 | 2 | 3 | Ham |
| Email 4 | 1 | 0 | 3 | 2 | Ham |

$$\theta_{ham,free} = Pr(free|ham) = (3+1)/(13+4)$$
$$\theta_{ham,bank} = Pr(bank|ham) = (0+1)/(13+4)$$
$$\theta_{ham,meet} = Pr(meet|ham) = (5+1)/(13+4)$$
$$\theta_{ham,time} = Pr(time|ham) = (5+1)/(13+4)$$
$$\pi_{ham} = Pr(ham) = 2/4$$

# Dealing with missing features: Solution 2

More generally, pretend you've seen each word $\alpha \geq 1$ extra times.

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 0 | 2 | 3 | Ham |
| Email 4 | 1 | 0 | 3 | 2 | Ham |

$$p(\text{funny\_word}|\text{spam}) = \frac{\#\text{of funny\_word in spam emails} + \alpha}{\#\text{of words in spam emails} + \alpha \times \#\text{of unique words}}$$

# Laplacian Smoothing: History and effect on ML estimate

## History: What is the prob. that the sun will rise tomorrow?

- Given a large sample of days with the rising sun, we still can not be completely sure that the sun will still rise tomorrow

$$\Pr(\text{sun rising tomorrow}|\text{it rose } t \text{ times}) = \frac{t+1}{t+2}$$

## Effect on the ML estimate

- Laplace smoothing biases the ML estimate
- Equivalent to performing MAP estimation with a Dirichlet (multi-variate Beta) prior
- As training data size grows, the effect of Laplacian smoothing disappears

# Summary

You should know

- what a Naive Bayes model is
- why it is 'naive'
- how this model can be used to classify spam vs ham emails
- Handle missing features via Laplace smoothing

# Moving forward

Examine the classification rule for naive Bayes

$$y^* = \arg\max_c \left( \log \pi_c + \sum_k x_k \log \theta_{ck} \right)$$

For binary classification, we thus determine the label based on the sign of

$$\log \pi_1 + \sum_k x_k \log \theta_{1k} - \left( \log \pi_2 + \sum_k x_k \log \theta_{2k} \right)$$

This is just a linear function of the features (word-counts) $\{x_k\}$

$$w_0 + \sum_k x_k w_k$$

where we "absorb" $w_0 = \log \pi_1 - \log \pi_2$ and $w_k = \log \theta_{1k} - \log \theta_{2k}$.

# Naive Bayes is a linear classifier

Fundamentally, what really matters is the decision boundary

$$w_0 + \sum_k x_k w_k$$

This motivates many new methods. One of them is logistic regression, to be discussed in next lecture.