A* - *Informed Search*

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(Thanks to past instructors and several book authors)

Uninformed Search Complexity

- N = Total number of states
- B = Average number of successors (branching factor)
- L = Length for start to goal with smallest number of steps
- Q = Average size of the priority queue
- $Lmax$ = Length of longest path from *START* to any state

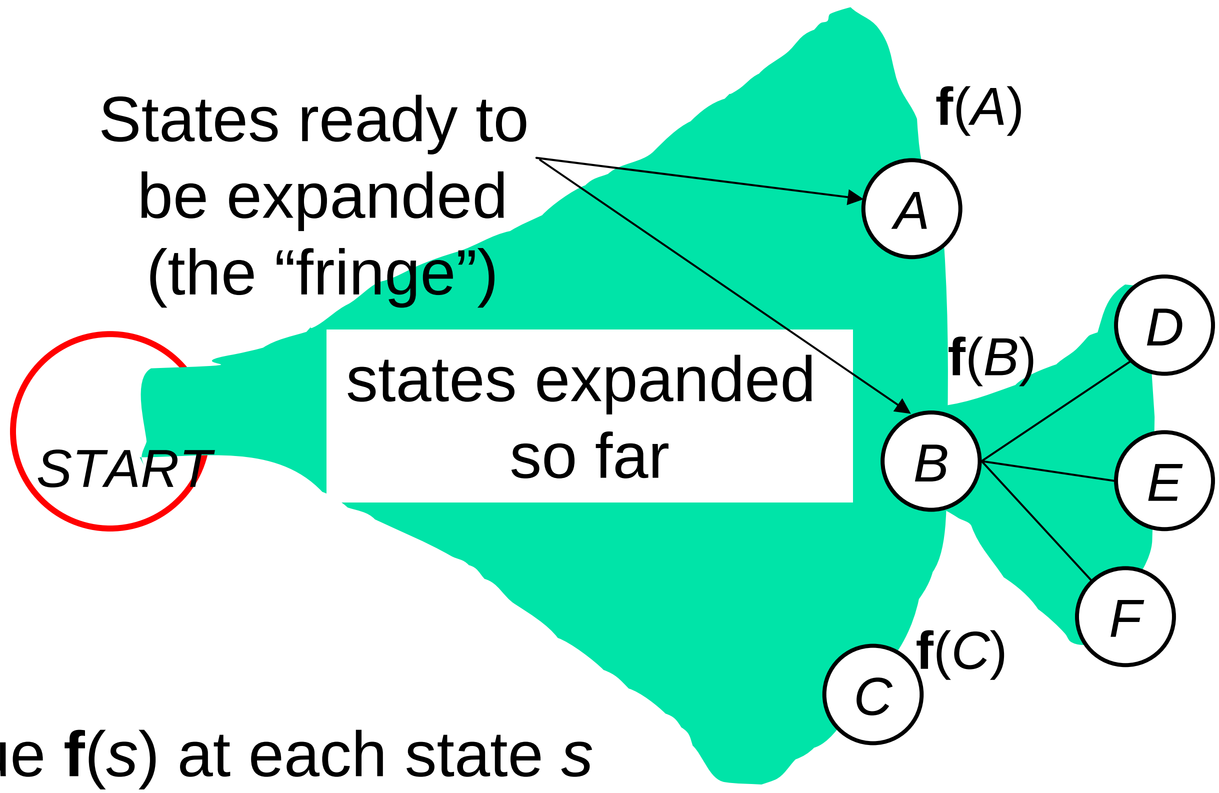
	Algorithm	Complete	Optimal	Time	Space
BFS	Breadth First Search	Y	Y, If all trans. have same cost	O(Min(N,B^L))	O(Min(N,B^L))
BIBFS	Bi- Direction. BFS	Y	Y, If all trans. have same cost	O(Min(N,2$B^{L/2}$))	O(Min(N,2$B^{L/2}$))
PCDFS	Path Check DFS	Y	N	O(B^{Lmax})	O(BL_{max})
MEMDFS	Memorizing DFS	Y	N	O(Min(N,B^{Lmax}))	O(Min(N,B^{Lmax}))
IDS	Iterative Deepening	Y	Y, If all trans. have same cost	O(B^L)	$O(BL)$

Uninformed vs Informed

- Uninformed – only guided by
  - s*uccessor* relationships
  - topological structure (leftmost,…)
  - length as number of nodes

- Informed
  - assume *cost* of edges
  - more knowledge?

Search Revisited

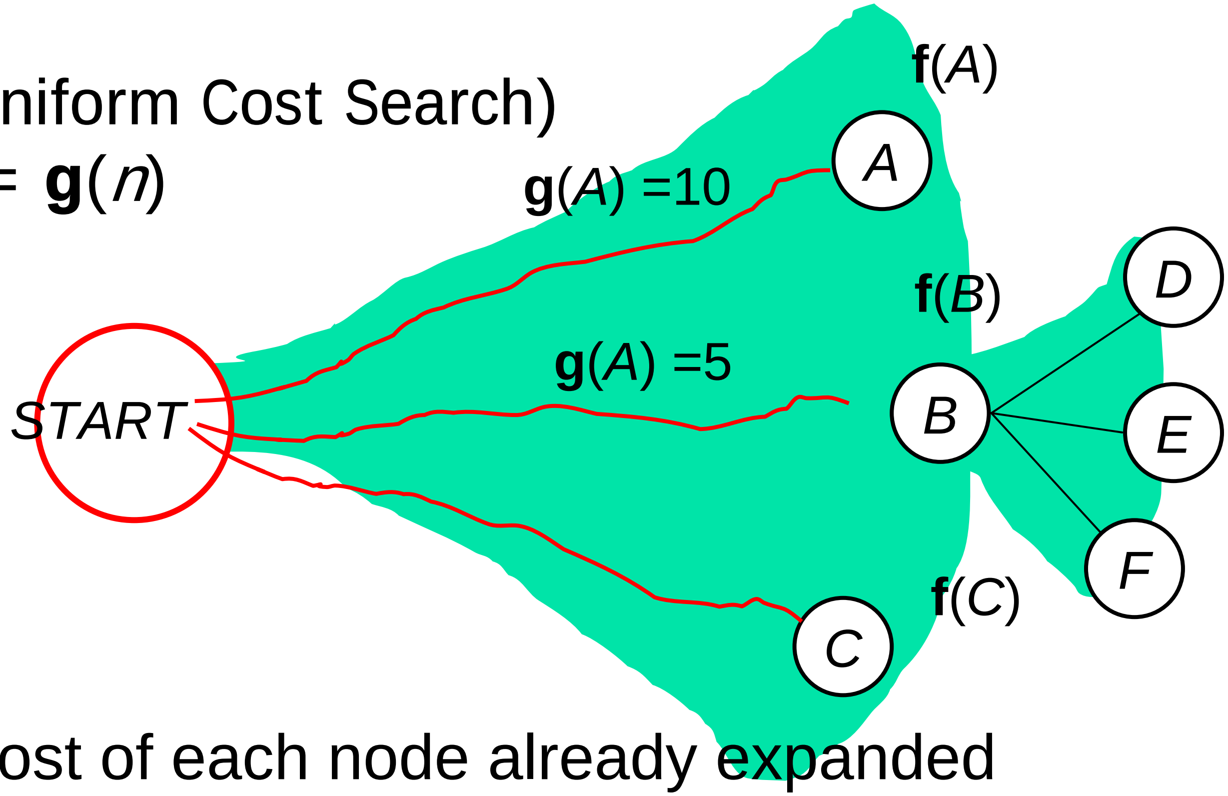


1. Store a value **f**(*s*) at each state *s*
2. Choose the state with lowest **f** to expand next
3. Insert its successors

*If **f**(.) is chosen carefully*, we will eventually find the lowest-cost sequence

• UCS (Uniform Cost Search)

f(n) = **g**(n)



• g(n) - cost of each node already expanded
length of shortest path from START to n

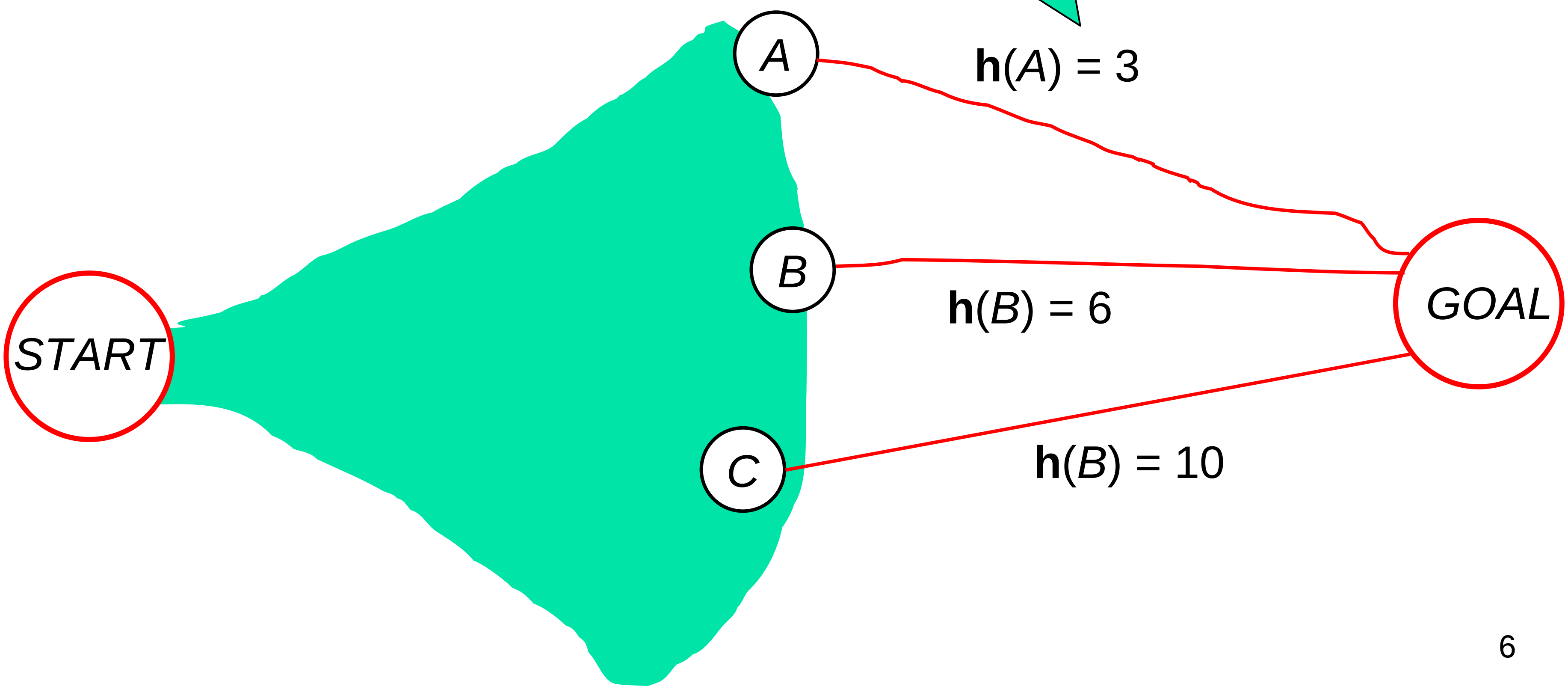
• Implementation – Store open successor states (waiting to be expanded) in a *priority queue* for efficient retrieval of minimum **f**

• Optimal → Guaranteed to find lowest cost sequence, *but guidance is about known path…*

Estimate "Cost" to Goal

- Introduce a function **h**(s) to estimate the unknown distance from state s to the goal

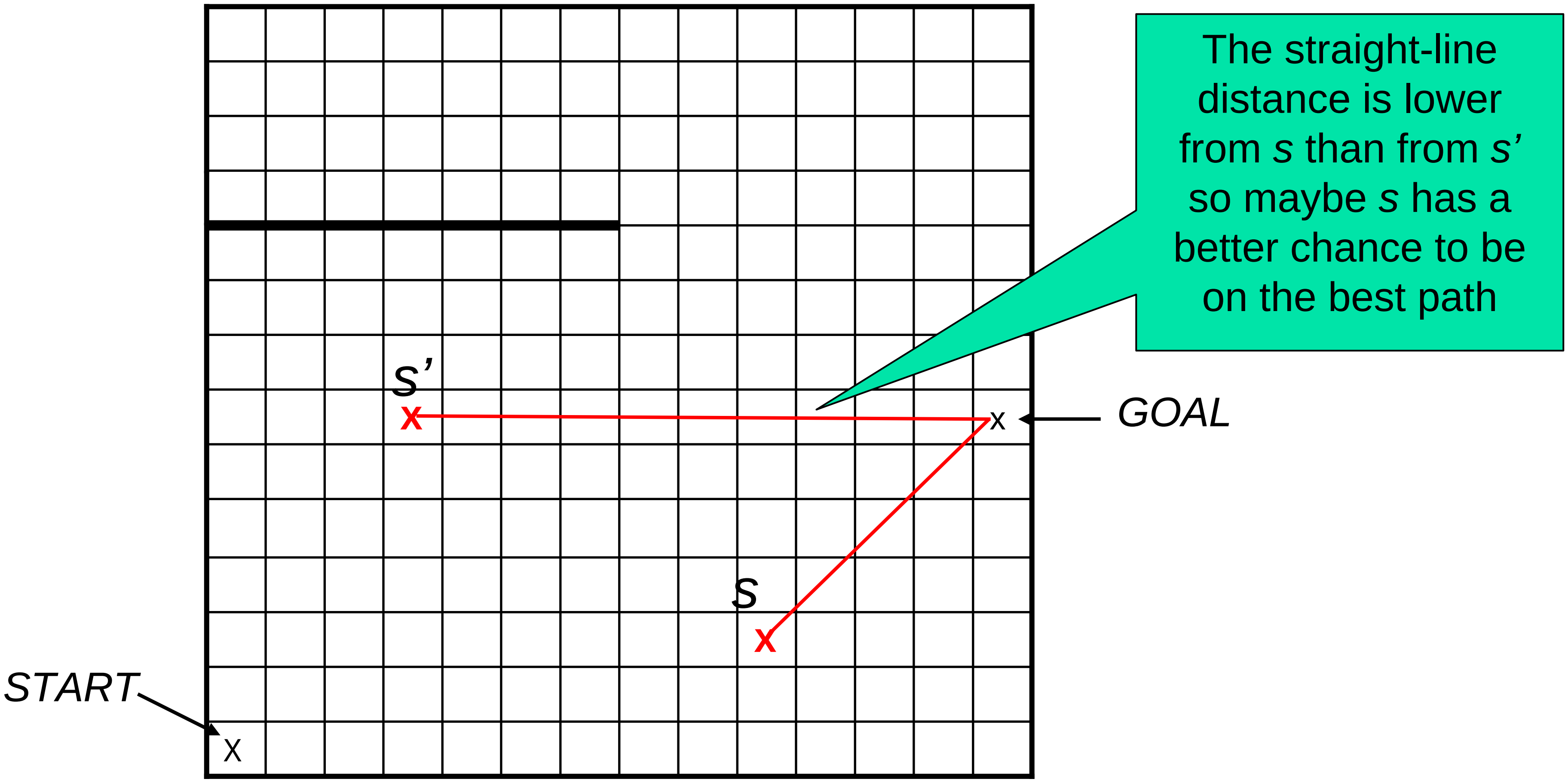
Our best **guess** is that *A* is closer to *GOAL* than *B* so maybe it is a more promising state to expand



Heuristic Functions

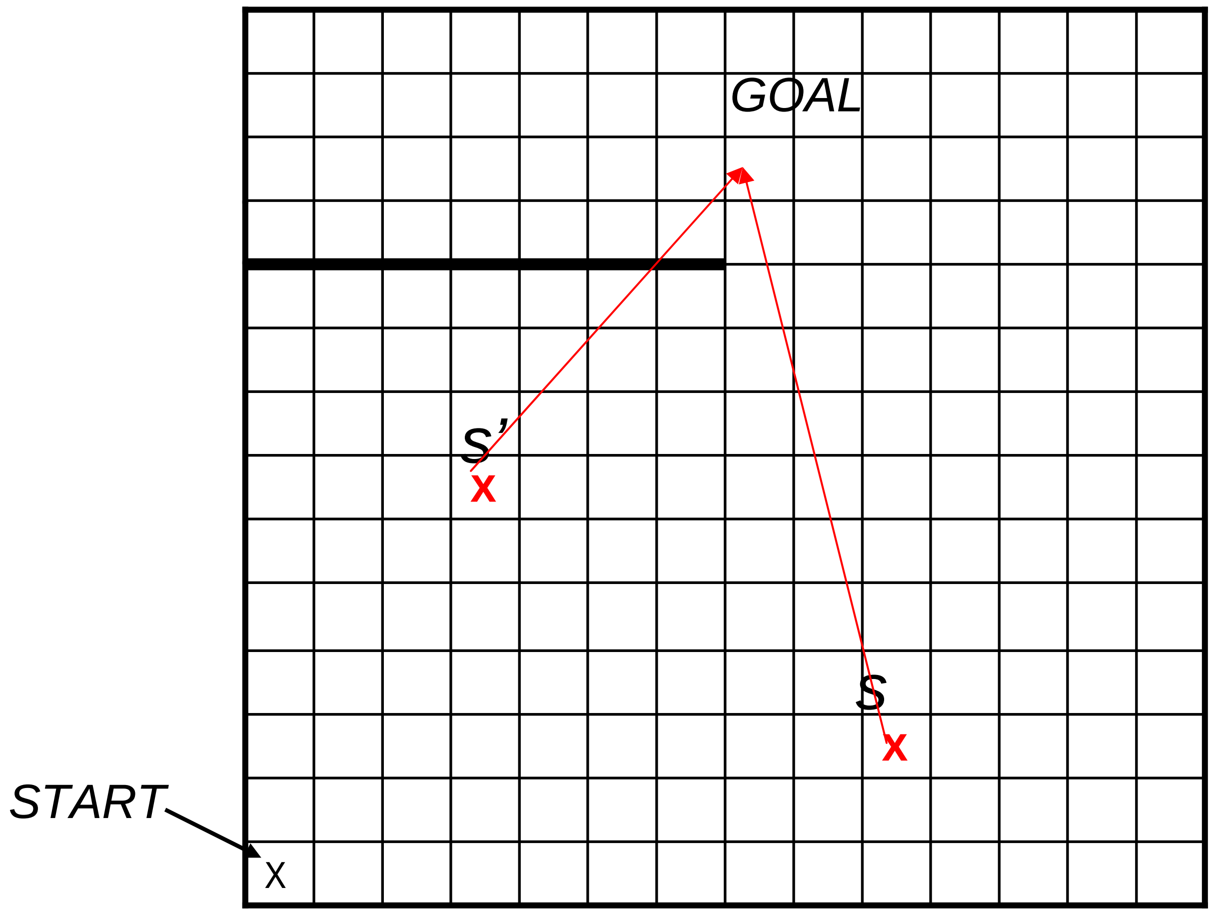
- **h** is a ***heuristic*** function for the search problem
- **h**(*s*) = estimate of the cost of the shortest path from *s* to GOAL
- **h** cannot be computed solely from the states and transitions in the current problem → If we could, we would already know the optimal path!
- **h**(.) is based on external knowledge about the problem → *informed* search
- Questions:
  1. Typical examples of **h**?
  2. How to use **h**?
  3. What are desirable/necessary properties of **h**?

Heuristic Functions Example



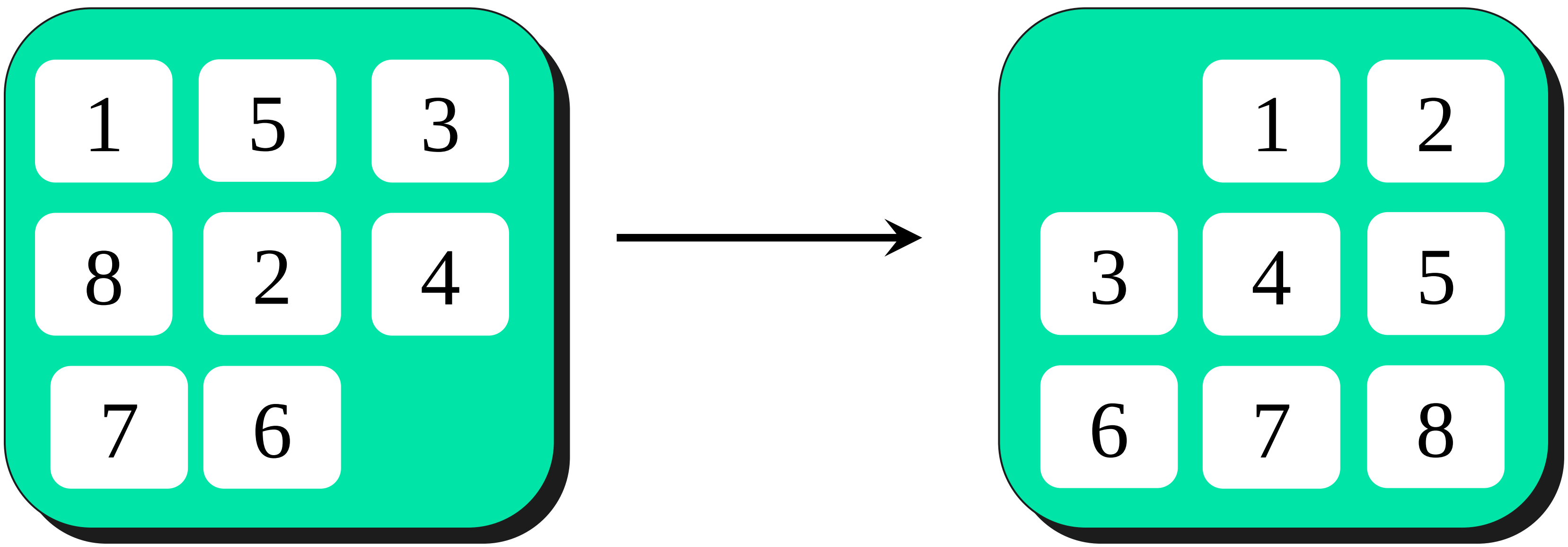
- **h**(*s*) = Euclidean distance to *GOAL*

Heuristic Functions Example



- **h**(*s*) = Euclidean distance to *GOAL*
- *Euclidean distance is an heuristic.*

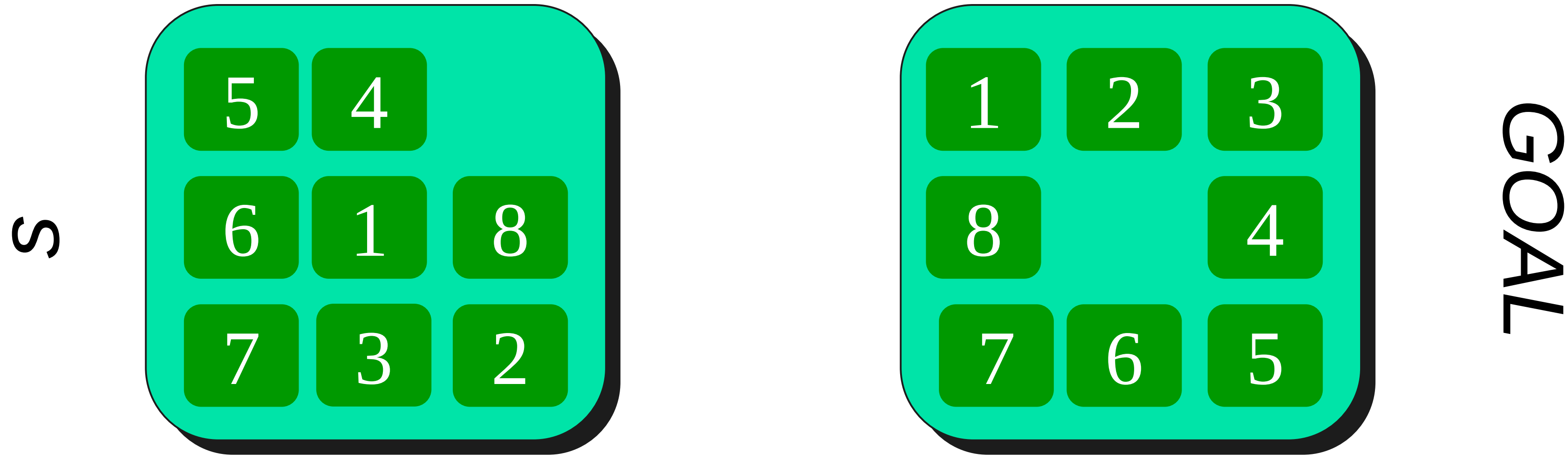
Heuristic Functions Example



s

GOAL

- How could we define **h**(*s*)?



Misplaced titles:

$\mathbf{h}_1(s) = 7$

Manhattan distance:

$\mathbf{h}_2(s) = 2 + 3 + 3 + 2 + 4 + 2 + 0 + 2 = 18$

First Attempt: Greedy Best First Search

- Simplest use of heuristic function: Always select the node with smallest **h**(.) for expansion (i.e., **f**(*s*) = **h**(*s*))

Initialize *PQ*

Insert *START* with value **h**(*START*) in *PQ*

While (*PQ* not empty and no goal state is in *PQ*)

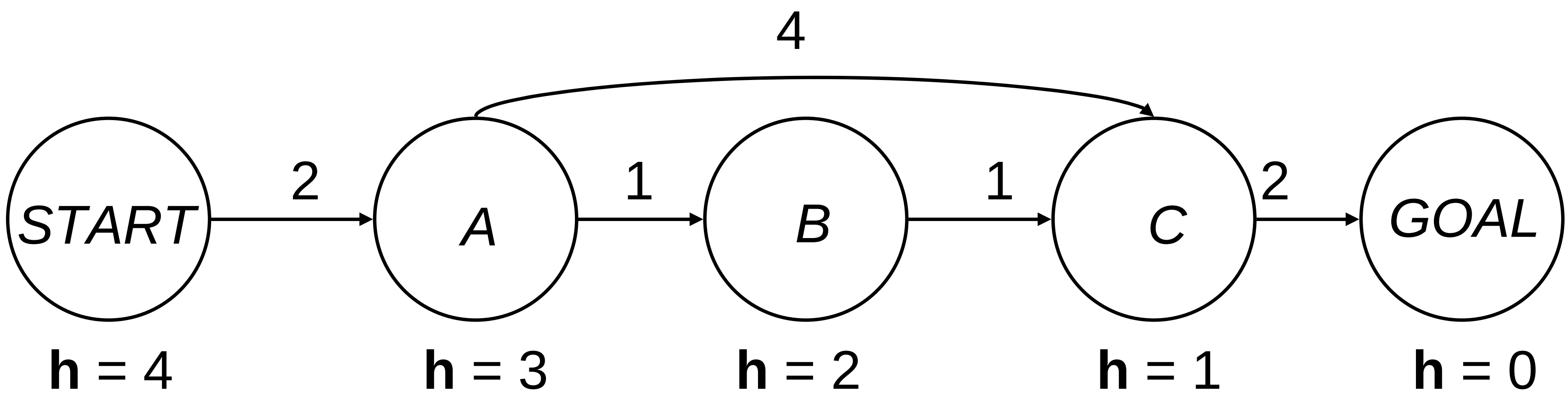
  Pop the state *s* with the minimum value of **h** from *PQ*

  For all *s*' in **succs**(*s*)

    If *s*' is not already in *PQ* and has not already been visited

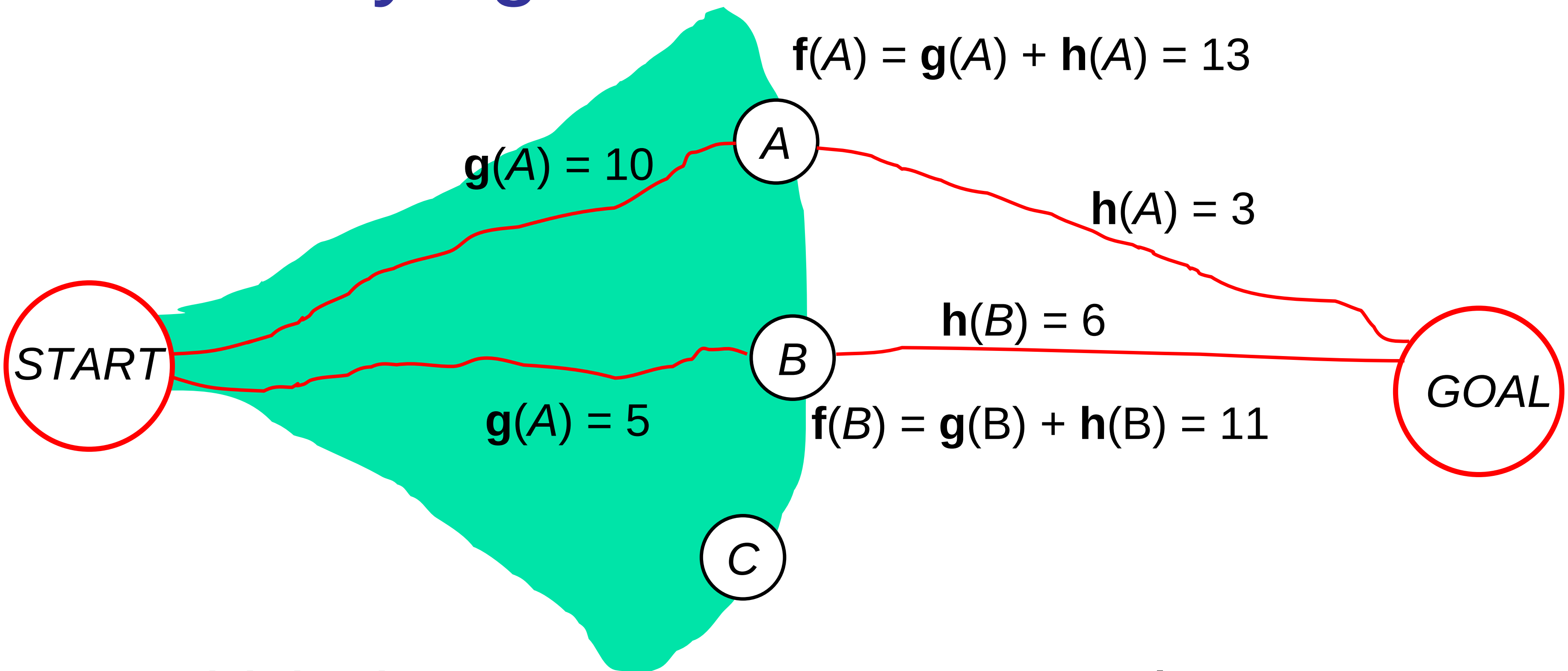
      Insert *s*' in *PQ* with value **h**(*s*')

Problem



- What solution do we find in this case?
- Greedy search clearly not optimal, even though the heuristic function is non-stupid

Trying to Fix the Problem



- **g**(*s*) is the cost from *START* to *s* only
- **h**(*s*) estimates the cost from *s* to *GOAL*
- Key insight: **g**(*s*) + **h**(*s*) estimates the ***total*** cost of the cheapest path from START to GOAL going through *s*
- → A* algorithm

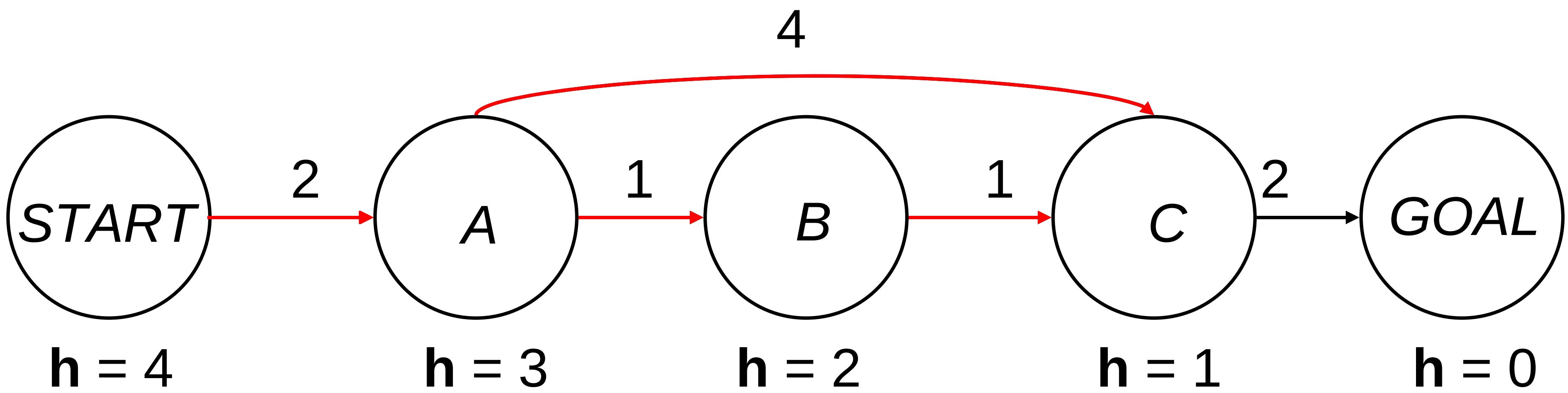
A* Algorithm

- **$f(s) = g(s) + h(s)$**

- heuristics
  - good, less good…, alternative, efficiency
  - “easy” to define…

- efficiency

Can A* Fix the Problem?



$\{(START,4)\}$

$\{(A,5)\}$

(**f**(A)=**h**(A)+**g**(A) = 3 + **g**($START$) + **cost**($START,A$) = 3 + 0 + 2)

$\{(B,5)\ (C,7)\}$

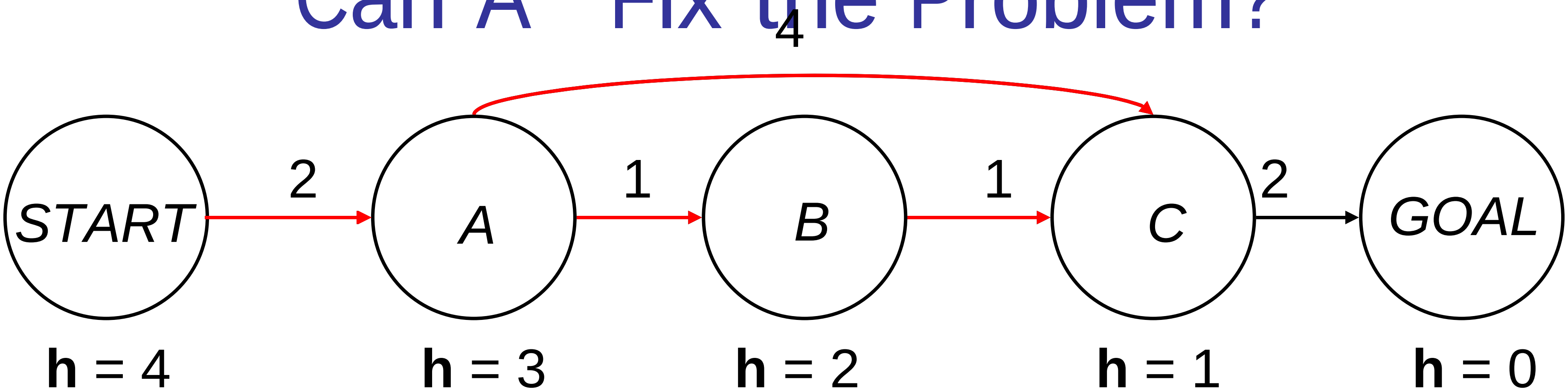
(**f**(C)=**h**(C)+**g**(C) = 1 + **g**(A) + **cost**(A,C) = 1 + 2 + 4)

$\{(C,5)\}$

(**f**(C)= **h**(C)+**g**(C) = 1 + **g**(B) + **cost**(B,C) = 1 + 3 + 1)

$\{(GOAL,6)\}$

Can A* Fix the Problem?



{(*START*,4)}

C is placed in the queue with backpointers {*A*,*START*}

{(*A*,5)}

(**f**(*A*)=**h**(*A*)+ **g**(*A*)= 3 + **g**(*START*) + **cost**(*START*,*A*) = 3 + 0 + 2)

{(*B*,5) (*C*,7)}

A lower value of **f**(*C*) is found with backpointers {*B*,*A*,*START*}

(**f**(*C*)=**h**(*C*) + **g**(*C*)= 1 + **g**(*A*) + **cost**(*A*,*C*)

{(*C*,5)}

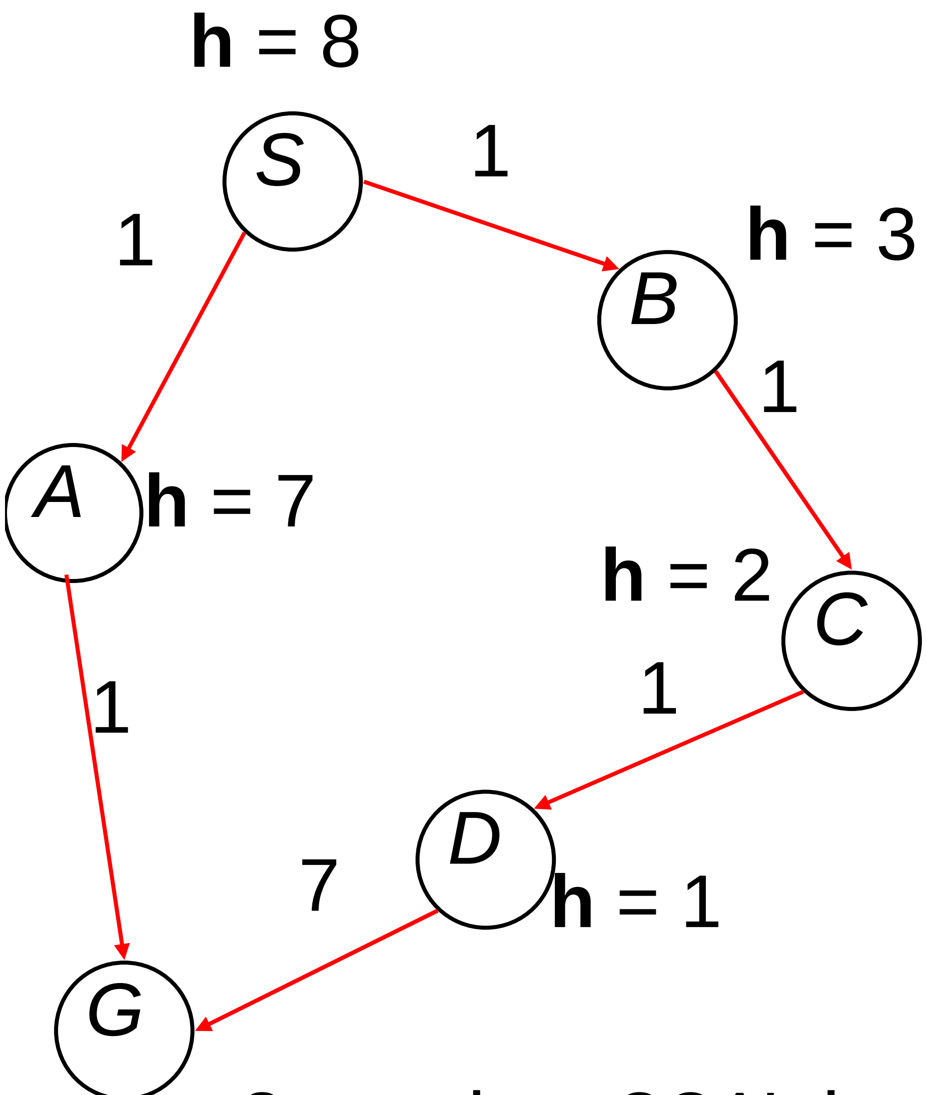
(**f**(*C*)=**h**(*C*) + **g**(*C*)= 1 + **g**(*B*) + **cost**(*B*,*C*) = 1 + 3 + 1)

{(*GOAL*,6)}

A* Core Issues

- Termination condition
- Revisiting states
- Algorithm
- Optimality
- Avoiding revisiting states
- Choosing good heuristics
- Reducing memory usage

A* Termination Condition



Queue:

{(B,4) (A,8)}

{(C,4) (A,8)}

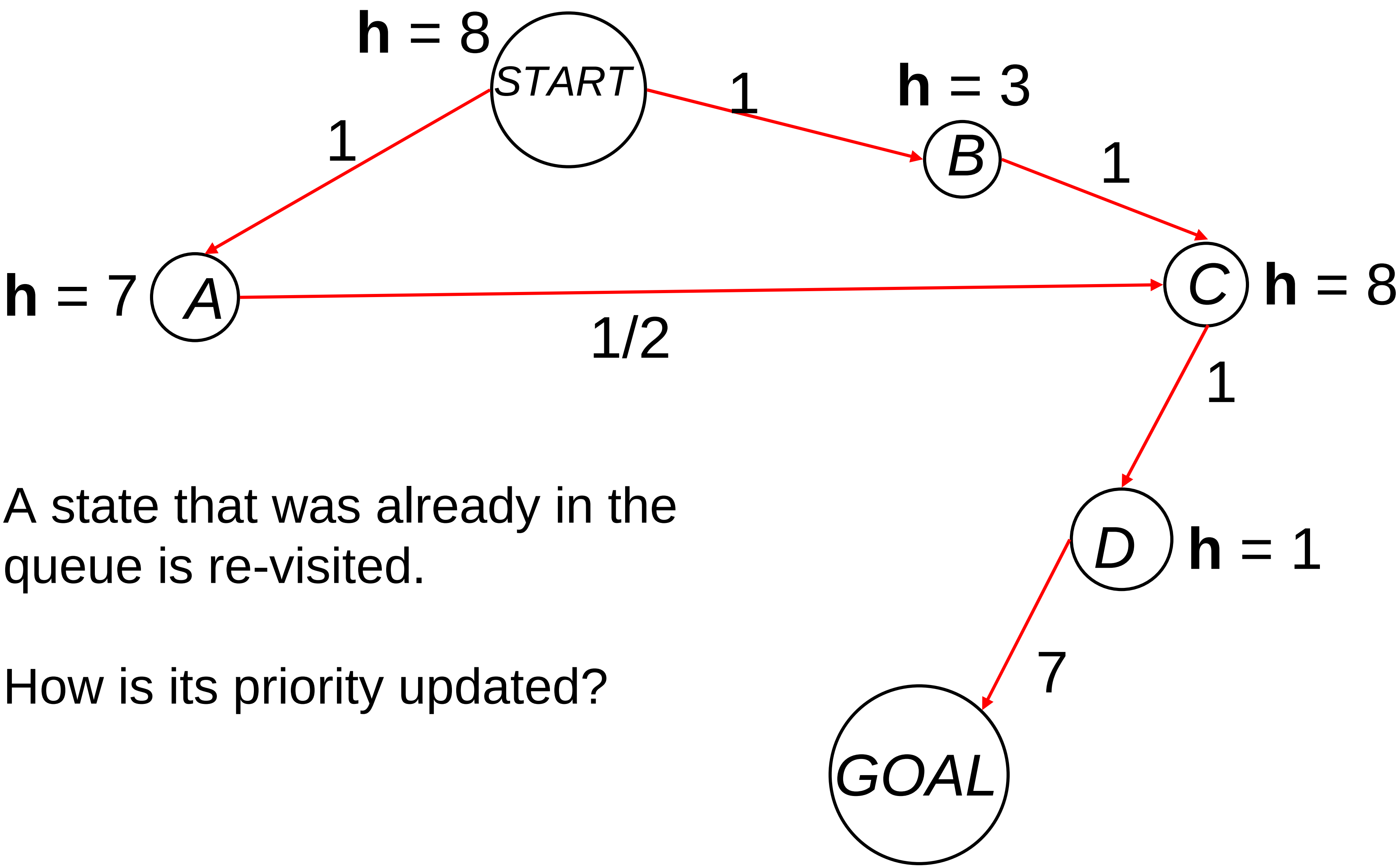
{(D,4) (A,8)}

{(A,8) (G,10)}

We have encountered *G* before we have a chance to visit the branch going through *A*. The problem is that at each step we use only an estimate of the path cost to the goal.

- Stop when GOAL is popped from the queue.

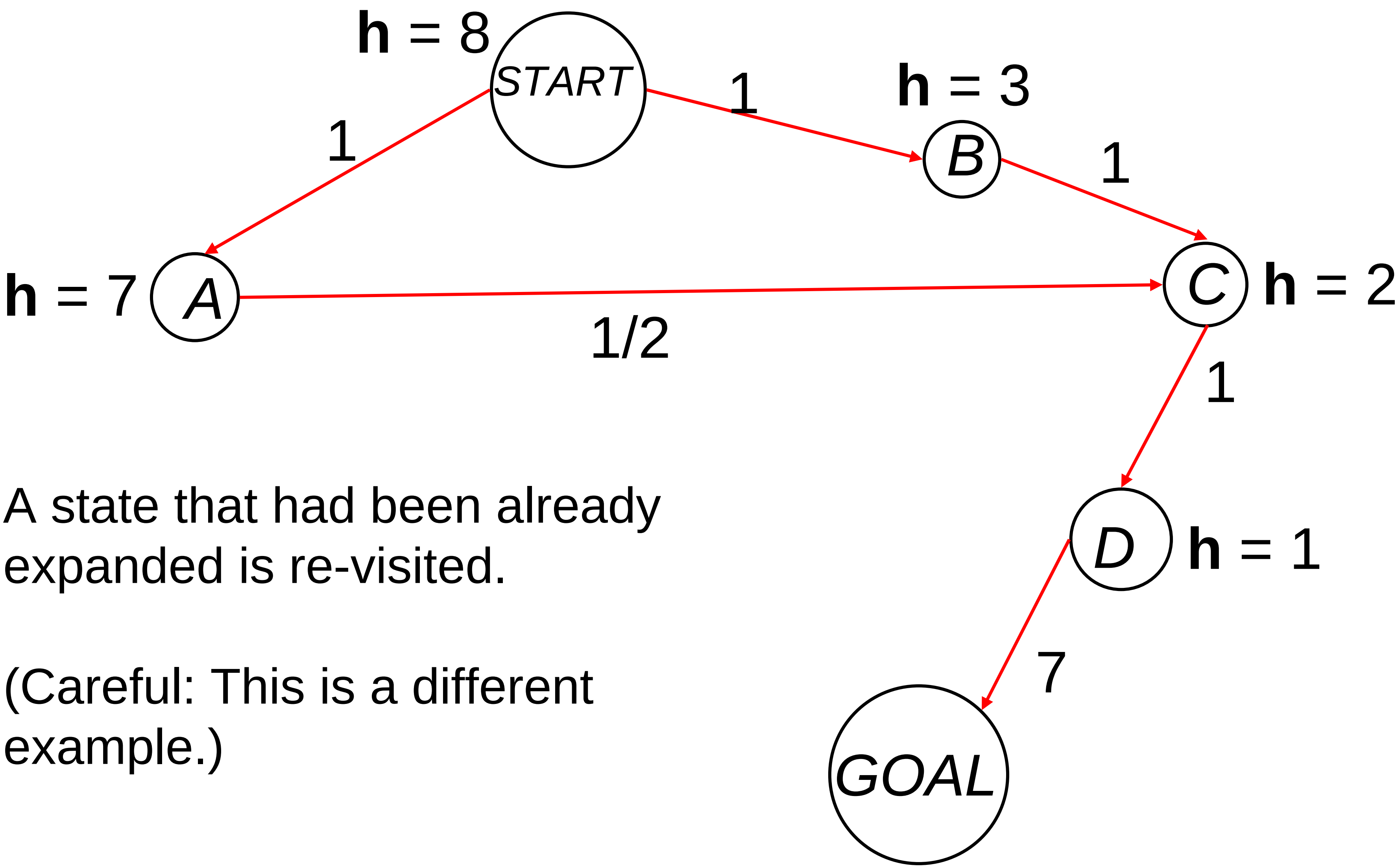
Revisiting States



A state that was already in the queue is re-visited.

How is its priority updated?

Revisiting States

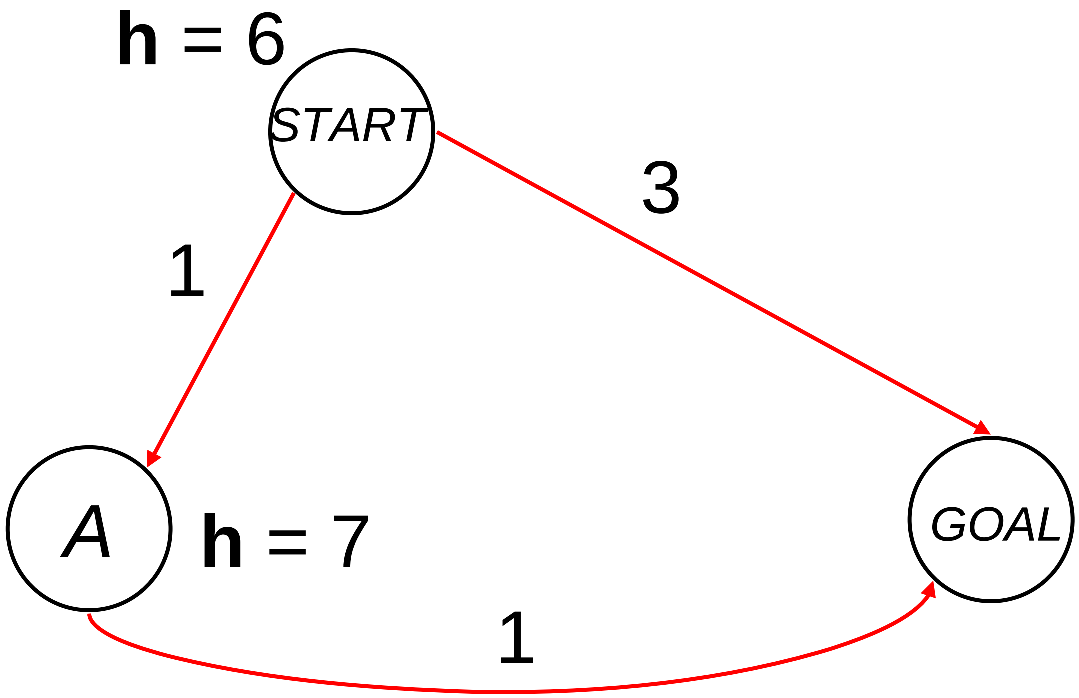


A state that had been already expanded is re-visited.

(Careful: This is a different example.)

Pop state *s* with lowest **f**(*s*) in queue
If *s* = *GOAL*
  return *SUCCESS*
Else expand *s*:
  For all *s'* in **succs** (*s*):
    f(*s*')= **g**(*s*') + **h**(*s'*) = **g**(*s*) + **cost**(*s*,*s'*) + **h**(*s'*)
    If (s' not seen before OR
      s' previously expanded with **f**(*s'*) > *f'* OR
      s' in PQ with with **f**(*s'*) > *f'*)
        Promote/Insert *s'* with new value *f'* in *PQ*
      previous(*s*') ← *s*
   Else Ignore *s*' (because it has been visited and its current path cost **f**(*s'*) is still the lowest path cost from *START* to *s*')

Under what Conditions is A* Optimal?



{(*START*,6)}
{(G*OAL*,3) (A,8)}

Final path:
{*START*, *GOAL*}
with cost = 3

- Problem: **h**(.) is a *poor* estimate of path cost to the goal state

Admissible Heuristics

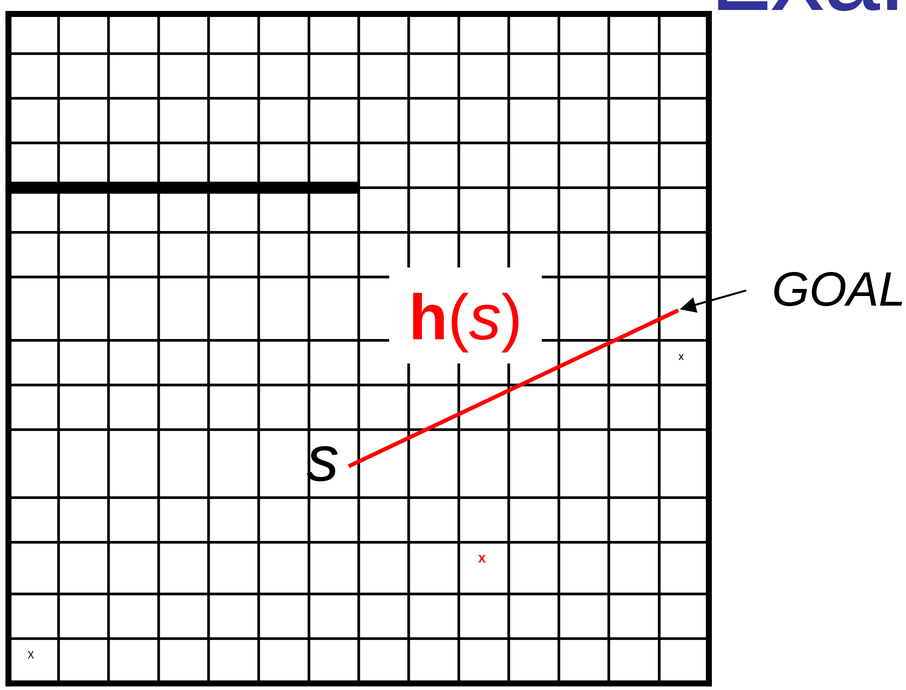
- Define $\mathbf{h}^*(s)$ = the true minimal cost to the goal from s
- $\mathbf{h}$ is admissible if

$$\boxed{\mathbf{h}(s) <= \mathbf{h}^*(s) \text{ for } \mathbf{all} \text{ states } s}$$

- I.e., an admissible heuristic never overestimates the cost to the goal. "Optimistic" estimate of cost to goal.

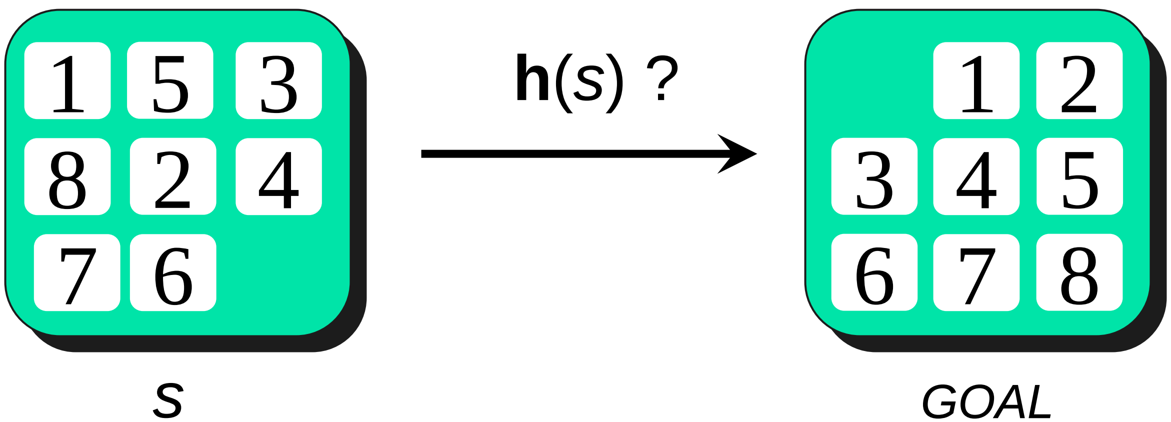
A* is guaranteed to find the optimal path if **h** is admissible. (proof in chapter 4)

Examples

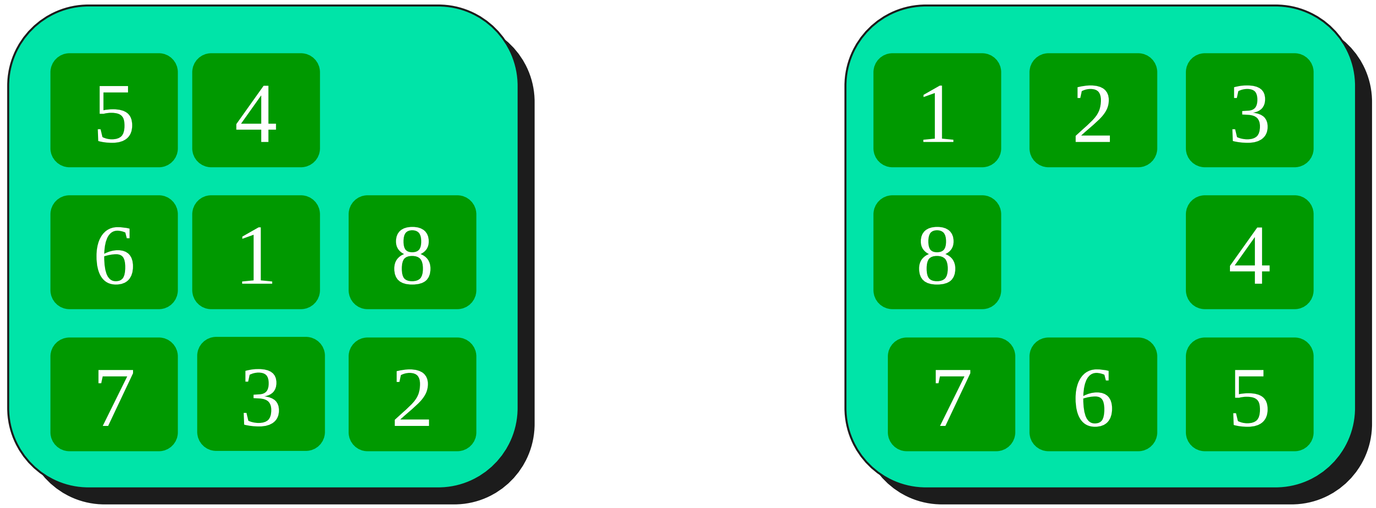


For the navigation problem: The length of the shortest path is at least the distance between *s* and *GOAL* → Euclidean distance is an admissible heuristic

What about the puzzle?



S



GOAL

Misplaced titles:

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Comparing Heuristics

$\mathbf{h}_1$ = misplaced tiles

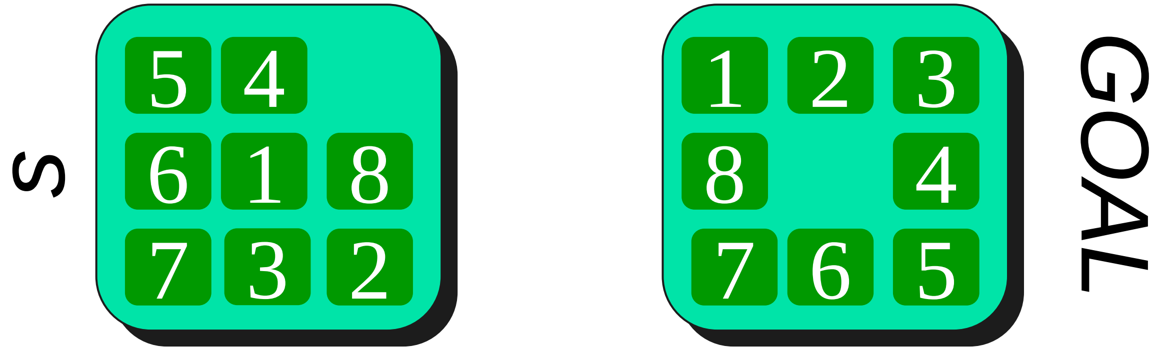
$\mathbf{h}_2$ = Manhattan distance

	L = 4 steps	L = 8 steps	L = 12 steps
Iterative Deepening	112	6,300	3.6×10^6
A* with heuristic $\mathbf{h}_1$	13	39	227
A* with heuristic $\mathbf{h}_2$	12	25	73

- Overestimates A* performance because of the tendency of IDS to expand states repeatedly
- Number of states expanded does not include log() time access to queue

Example from Russell&Norvig

Comparing Heuristics



$\mathbf{h}_1(s) = 7$

$\mathbf{h}_2(s) = 2 + 3 + 3 + 2 + 4 + 2 + 0 + 2 = 18$

$\mathbf{h}_2$ is larger than $\mathbf{h}_1$ and, at same time, A* seems to be more efficient with $\mathbf{h}_2$.

$\mathbf{h}_2$ *dominates* $\mathbf{h}_1$, if $\mathbf{h}_2(s) >= \mathbf{h}_1(s)$ for all s

For any two heuristics $\mathbf{h}_2$ and $\mathbf{h}_1$:
If $\mathbf{h}_2$ dominates $\mathbf{h}_1$ then A* is more efficient (expands fewer states) with $\mathbf{h}_2$

Intuition: since **h** <= **h***, a larger **h** is a better approximation of the true path cost

Limitations

- Computation: In the worst case, we may have to explore all the states

- The good news: A* is optimally efficient → For a given **h**(.), no other optimal algorithm will expand fewer nodes

- The bad news: Storage is also potentially exponential (all states)

IDS (Iterative Deepening Search)

- Need to make DFS optimal

- IDS (Iterative Deepening Search):
  - Run DFS by searching only path of length 1 (DFS stops if length of path is greater than 1)
  - If that doesn't find a solution, try again by running DFS on paths of length 2 or less
  - If that doesn't find a solution, try again by running DFS on paths of length 3 or less
  - ………..
  - Continue until a solution is found

Example: IDA* (Iterative Deepening A*)

- Same idea as Iterative Deepening DFS except use **f**(*s*) to control depth of search instead of the number of transitions
- Example, assuming integer costs:

1. Run DFS, stopping at states *s* such that <u>**f**(*s*) > 0</u>
   Stop if goal reached
2. Run DFS, stopping at states *s* such that <u>**f**(*s*) > 1</u>
   Stop if goal reached
3. Run DFS, stopping at states *s* such that <u>**f**(*s*) > 2</u>
   Stop if goal reached

……..Keep going by increasing the limit on **f** by 1 every time

- Complete (assuming we use loop-avoiding DFS)
- Optimal
- More expensive in computation cost than A*
- Memory order *L* as in DFS

Summary

- Informed search and heuristics
- First attempt: Best-First Greedy search
- A* algorithm
  - Optimality
  - Condition on heuristic functions
  - Completeness, efficiency
- IDA*

Nils Nilsson. Problem Solving Methods in Artificial Intelligence. McGraw Hill (1971)
Judea Pearl. Heuristics: Intelligent Search Strategies for Computer Problem Solving (1984)
Chapters 3&4 Russell & Norvig