15-381: AI
Markov Systems and Markov Decision Processes

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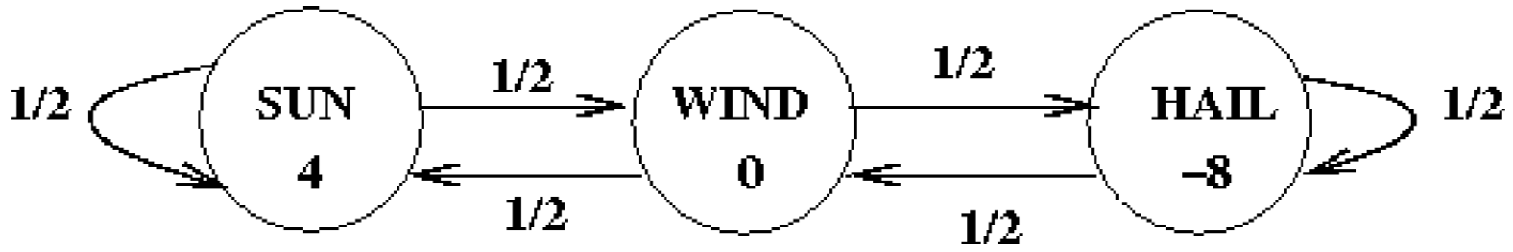
States, Actions, Observations

	Passive	Controlled
Fully Observable	**Markov Model**	**Markov Decision Process (MDP)**
Hidden State	**Hidden Markov Model - HMM**	**Partially Observable MDP (POMDP)**

Markov Model – Observation



- States
- Rewards
- Transition probabilities

Markov Systems with Rewards

- Finite set of n states, s_i
- Probabilistic state matrix, P, p_{ij}
  - Probabilty of transition from s_i to sj
- "Goal achievement" - Reward for each state, r_i
- Discount factor - γ

"Solving" Markov Systems with Rewards

- Find "value of each state"
- Process:
  - Start at state s_i
  - Receive immediate reward r_i
  - Move randomly to a new state according to the probability transition matrix
  - Future rewards (of next state) are discounted by γ

Solving a Markov System with Rewards

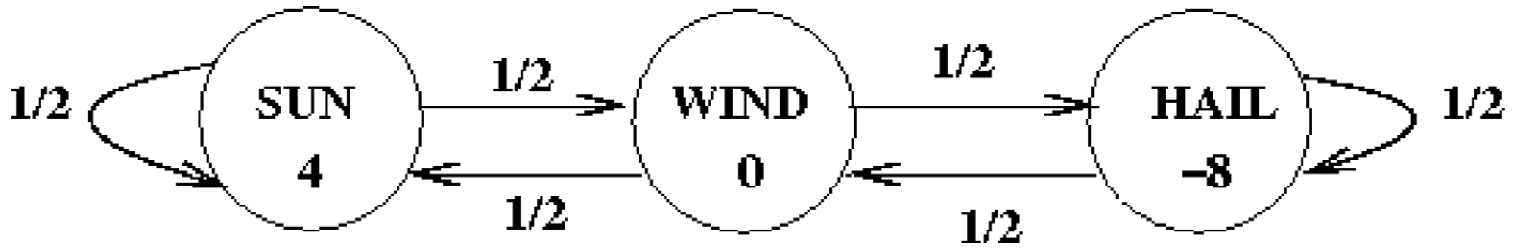
- $V^*(s_i)$ - expected discounted sum of future rewards starting in state s_i

- $V^*(s_i) = r_i + \gamma[p_{i1}V^*(s_1) + p_{i2}V^*(s_2) + \ldots p_{in}V^*(s_n)]$

Value Iteration to Solve a Markov System with Rewards

- $V^1(s_i)$ - expected discounted sum of future rewards starting in state s_i *for one step*.
- $V^2(s_i)$ - expected discounted sum of future rewards starting in state s_i *for two steps*.
- …
- $V^k(s_i)$ - expected discounted sum of future rewards starting in state s_i *for* k *steps*.
- As $k \to \infty V^k(s_i) \to V^*(s_i)$
- Stop when difference of k + 1 and k values is smaller than some $\in$.

3-State Example



3-State Example: Values $\gamma = 0.5$

Iteration	SUN	WIND	HAIL
0	0	0	0
1	4	0	-8
2	5.0	-1.0	-10.0
3	5.0	-1.25	-10.75
4	4.9375	-1.4375	-11.0
5	4.875	-1.515625	-11.109375
6	4.8398437	-1.5585937	-11.15625
7	4.8203125	-1.5791016	-11.178711
8	4.8103027	-1.5895996	-11.189453
9	4.805176	-1.5947876	-11.194763
10	4.802597	-1.5973969	-11.197388
11	4.8013	-1.5986977	-11.198696
12	4.8006506	-1.599349	-11.199348
13	4.8003254	-1.5996745	-11.199675
14	4.800163	-1.5998373	-11.199837
15	4.8000813	-1.5999185	-11.199919

3-State Example: Values $\gamma = 0.9$

Iteration	SUN	WIND	HAIL
0	0	0	0
1	4	0	-8
2	5.8	-1.8	-11.6
3	5.8	-2.6100001	-14.030001
4	5.4355	-3.7035	-15.488001
5	4.7794	-4.5236254	-16.636175
6	4.1150985	-5.335549	-17.521912
7	3.4507973	-6.0330653	-18.285858
8	2.8379793	-6.6757774	-18.943516
9	2.272991	-7.247492	-19.528683
...	...	...	...
50	-2.8152928	-12.345073	-24.633476
51	-2.8221645	-12.351946	-24.640347
52	-2.8283496	-12.3581295	-24.646532
...	...	...	...
86	-2.882461	-12.412242	-24.700644
87	-2.882616	-12.412397	-24.700798
88	-2.8827558	-12.412536	-24.70094

3-State Example: Values $\gamma = 0.2$

Iteration	SUN	WIND	HAIL
0	0	0	0
1	4	0	-8
2	4.4	-0.4	-8.8
3	4.4	-0.44000003	-8.92
4	4.396	-0.452	-8.936
5	4.3944	-0.454	-8.9388
6	4.39404	-0.45443997	-8.93928
7	4.39396	-0.45452395	-8.939372
8	4.393944	-0.4545412	-8.939389
9	4.3939404	-0.45454454	-8.939393
10	4.3939395	-0.45454526	-8.939394
11	4.3939395	-0.45454547	-8.939394
12	4.3939395	-0.45454547	-8.939394

Markov Decision Processes

- Finite set of states, $s_1,\ldots, s_n$
- Finite set of actions, $a_1,\ldots, a_m$
- Probabilistic state,action transitions:

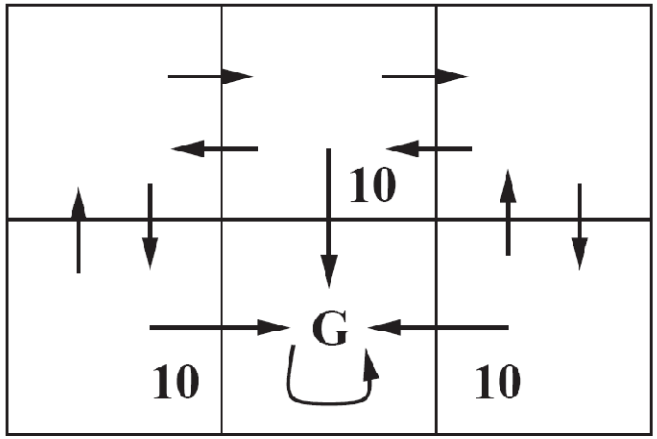
$$p_{ij}^{k} = \text{prob}\,(\text{next} = s_j \,\big|\, \text{current} = s_i \text{ and take action } a_k)$$

- Reward for each state, $r_1,\ldots, r_n$

Solving an MDP

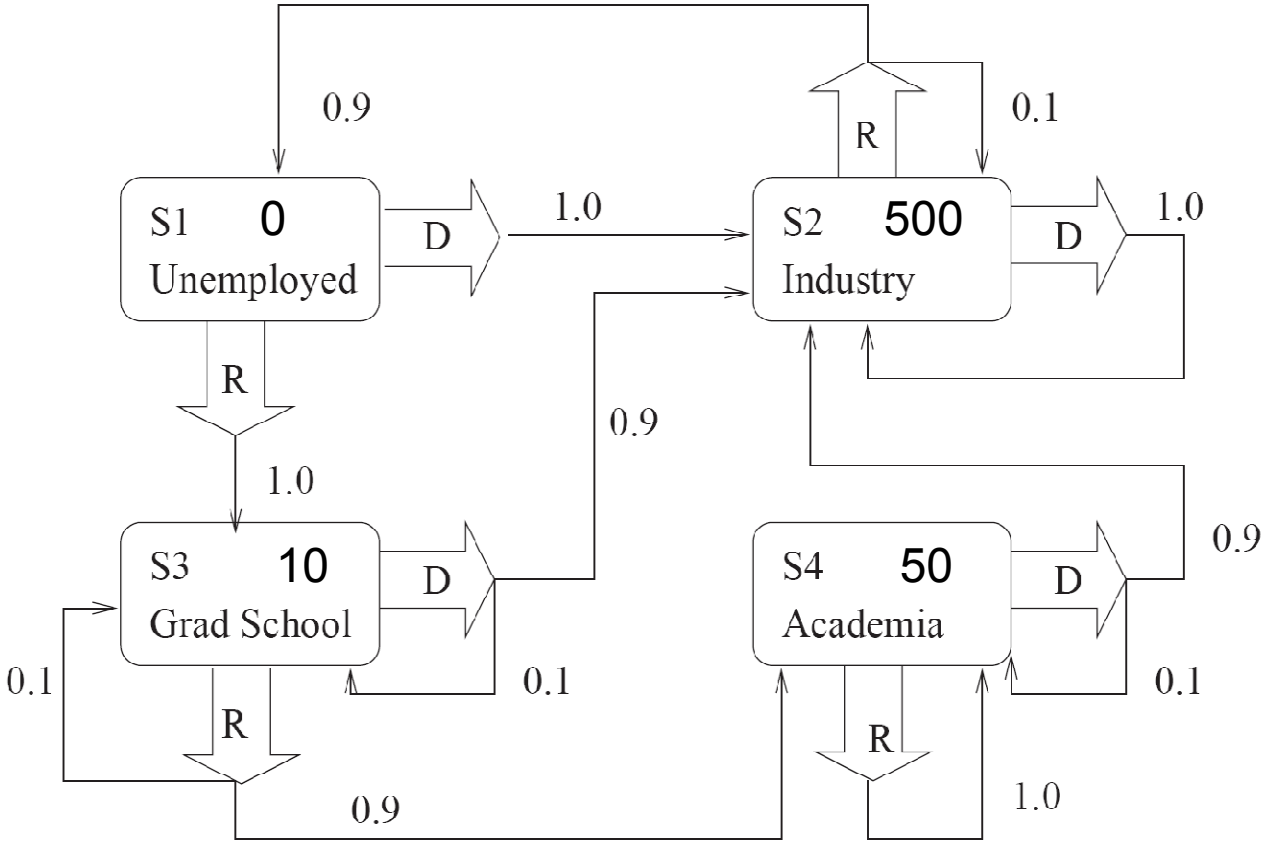
- A policy is a mapping from states to actions.
- Optimal policy - for every state, there is no other action that gets a higher sum of discounted future rewards.
- For every MDP there exists an optimal policy.
- Solving an MDP is finding an optimal policy.
- A specific policy converts an MDP into a plain Markov system with rewards.

Deterministic MDP Example



(Reward on unlabelled transitions is zero.)

Nondeterministic Example



Solving MDPs

- Find the value of each state and the action to take from each state.
- Process:
  - Start in state s_i
  - Receive immediate reward r_i
  - Choose action $a_k \in A$
  - Change to state s_j with probability
  - Discount future rewards

Policy Iteration

- Start with some policy $\pi_0(s_i)$.
- Such policy transforms the MDP into a plain Markov system with rewards.
- Compute the values of the states according to the current policy.
- Update policy:

$$\pi_1(s_i) = \arg\max_a \{r_i + \gamma \sum_j p_{ij}^a V^{\pi 0}(s_j)\}$$

- Keep computing
- Stop when $\pi_{k+1} = \pi_k$.

Value Iteration

- $V^*(s_i)$ - expected discounted future rewards, if we start from state s_i, and we follow the optimal policy.
- Compute V^* with value iteration:
  - $V^k(s_i)$ = maximum possible future sum of rewards starting from state s_i for k steps.
- Bellman's Equation:

$$V^{n+1}(s_i) = \max_k \{r_i + \gamma \sum_{j=1}^{N} p_{ij}^k V^n(s_j)\}$$

- Dynamic programming

Summary

- Markov model for state/action transitions.
- Markov systems with reward - goal achievement
- Markov decision processes - added actions
- Value, policy iteration