Symbolic Integration

Victor Adamchik

Carnegie Mellon University

Resultant

The origin of the resultant lies in papers by Euler (1748) and Bezout (1764) for determining when two polynomials have a non-trivial common factor. However, we will follow the elegant derivation by Sylvester (1840). Let us start with a simple example.

Consider a system of quadratic equations

$$\begin{cases} f = a_2 x^2 + a_1 x + a_0 = 0 \\ \\ g = b_2 x^2 + b_1 x + b_0 = 0 \end{cases}$$

If it has a common solution, f and g must have a common factor h

$$f = q_1 * h$$
$$g = q_2 * h$$

We can write

$$f * q_2 = g * q_1$$

or

$$\left(a_2 x^2 + a_1 x + a_0\right)(c_1 x + c_0) = \left(b_2 x^2 + b_1 x + b_0\right)(-d_1 x - d_0)$$

where coefficients c_k and d_k are unknown. Expand it and then collect terms wrt x

$$(a_2 c_1 + b_2 d_1) x^3 + (a_2 c_0 + a_1 c_1 + b_2 d_0 + b_1 d_1) x^2 + (a_1 c_0 + a_0 c_1 + b_1 d_0 + b_0 d_1) x + a_0 c_0 + b_0 d_0$$

Next, we equal coefficient by x to zero to get the following system

$$\begin{aligned} a_2 c_1 + b_2 d_1 &= 0 \\ a_2 c_0 + a_1 c_1 + b_2 d_0 + b_1 d_1 &= 0 \\ a_1 c_0 + a_0 c_1 + b_1 d_0 + b_0 d_1 &= 0 \\ a_0 c_0 + b_0 d_0 &= 0 \end{aligned}$$

or

$$0\ c_0 + a_2\ c_1 + 0\ d_0 + b_2\ d_1 = 0$$
$$a_2\ c_0 + a_1\ c_1 + b_2\ d_0 + b_1\ d_1 = 0$$
$$a_1\ c_0 + a_0\ c_1 + b_1\ d_0 + b_0\ d_1 = 0$$
$$a_0\ c_0 + 0\ c_1 + b_0\ d_0 + 0\ d_1 = 0$$

or, in matrix form

$$\begin{pmatrix} 0 & a_2 & 0 & b_2 \\ a_2 & a_1 & b_2 & b_1 \\ a_1 & a_0 & b_1 & b_0 \\ a_0 & 0 & b_0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ d_0 \\ d_1 \end{pmatrix} = 0$$

The system has a non-trivial solution when the determinate is zero

$$\begin{vmatrix} 0 & a_2 & 0 & b_2 \\ a_2 & a_1 & b_2 & b_1 \\ a_1 & a_0 & b_1 & b_0 \\ a_0 & 0 & b_0 & 0 \end{vmatrix} = 0$$

By transposition

$$\begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & 0 \\ 0 & b_2 & b_1 & b_0 \end{vmatrix} = 0$$

This determinate is called the Sylvester resultant.

General case. Let

$$f(x) = \sum_{k=0}^{m} a_k\, x^k \ \text{ and } \ g(x) = \sum_{k=0}^{n} b_k\, x^k$$

be non-constant polynomials.

Definition. The *Sylvester* matrix of f and g is an $(m+n)$ x$(m+n)$ matrix of coefficients defined by (assuming $m \geq n$)

$$\begin{pmatrix} a_m & a_{m-1} & \ldots & \ldots & a_{m-n+1} & a_{m-n} & \ldots & \ldots & a_0 & \ldots & \ldots & 0 \\ 0 & a_m & a_{m-1} & \ldots & \ldots & a_{m-n+1} & \ldots & \ldots & a_1 & a_0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & a_m & a_{m-1} & \ldots & \ldots & a_{n-1} & \ldots & a_1 & a_0 \\ b_n & b_{n-1} & \ldots & \ldots & b_1 & b_0 & 0 & \ldots & 0. & \ldots & \ldots & 0 \\ 0 & b_n & b_{n-1} & \ldots & \ldots & b_1 & b_0 & \ldots & 0. & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & b_{n-1} & \ldots & b_1 & b_0 \end{pmatrix}$$

Observation.

The upper part (that involves only a_k) has $n = \deg(g)$ rows.

The bottom part (that involves only b_k) has $m = \deg(f)$ rows.

Example. Consider

$$f(x) = x^4 - 3\,x^3 + 2\,x + 1$$

$$g(x) = x^3 - 1$$

The Sylvester matrix is

$$\begin{pmatrix} 1 & -3 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 2 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

Definition. *The resultant* $res_x(f, g)$ *of two univariate polynomials* f *and* g *over an intergral domain (a commutative ring with identity having no zero-divisors) is the determinant of the Sylvester matrix.*

For the above example, the determinate is

$$\mathbf{Det}\left[\begin{pmatrix} 1 & -3 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 2 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}\right]$$

```
19
```

and resultant is

$$\mathbf{Resultant}\left[\mathbf{x^4 - 3\,x^3 + 2\,x + 1,\ x^3 - 1,\ x}\right]$$

```
19
```

Exercise. Write the Sylvester matrix for

$$f(x) = x^4 - 3\,x^3 + 2\,x + 1$$

$$g(x) = 1$$

We have showed so far

$$\text{res}_x(f, g) = 0 \implies \det(S) = 0$$

Theorem.

Two polynomials f and g have non-trivial common factors $\Longleftrightarrow$ *$res_x(f, g) = 0$.*

Proof. Let

$$f(x) = \sum_{k=0}^{m} a_k x^k \text{ and } g(x) = \sum_{k=0}^{n} b_k x^k$$

be non-constant polynomials and $m \geq n$. Then equation

$$u * f + v * g = 0$$

$$\deg(u) < \deg(g)$$

$$\deg(v) < \deg(f)$$

has a solution iff f and g have common factors. Rewrite the above equation in a polynomial form

$$\left(u_{n-1} x^{n-1} + \ldots + u_0\right)(a_m x^m + \ldots + a_0) + \left(v_{m-1} x^{m-1} + \ldots + v_0\right)(b_n x^n + \ldots + b_0) = 0$$

Expanding it and then equating coefficients by x to zero

$$x^{m+n-1}: \quad u_{n-1} a_m + v_{m-1} b_n$$

$$x^{m+n-2}: \quad u_{n-1} a_{m-1} + u_{n-2} a_m + v_{m-1} b_{m-1} + v_{m-2} b_m$$

$$x^{n-1}: \quad u_{n-1} a_0 + u_{n-2} a_1 + \ldots + u_0 a_{n-1} + v_0 b_{n-1} + v_1 b_{n-2}$$

gives the sytem of equation that can be written in a matrix form as follows

$$(u_{n-1}, \ldots, u_0, v_{m-1}, \ldots, v_0) \begin{pmatrix} a_m & a_{m-1} & \ldots & \ldots & a_{m-n+1} & a_{m-n} & \ldots & \ldots & a_0 & \ldots & \ldots & 0 \\ 0 & a_m & a_{m-1} & \ldots & \ldots & a_{m-n+1} & \ldots & \ldots & a_1 & a_0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & a_m & a_{m-1} & \ldots & \ldots & a_{n-1} & \ldots & a_1 & a_0 \\ b_n & b_{n-1} & \ldots & \ldots & b_1 & b_0 & 0 & \ldots & 0. & \ldots & \ldots & 0 \\ 0 & b_n & b_{n-1} & \ldots & \ldots & b_1 & b_0 & \ldots & 0. & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & b_{n-1} & \ldots & b_1 & b_0 \end{pmatrix} = 0$$

The system $V . S$ has a non-trivial solution when $\det(S) = 0$. QED.

Example.

```
Resultant[x^5 - 3 x^4 + 2 x^3 - 4 x^2 + 4, x^5 + 2 x^3 - x^2 - 2, x]
```

```
0
```

If two polynomials indeed have common roots, they can be found by a GCD computation

```
PolynomialGCD[x^5 - 3 x^4 + 2 x^3 - 4 x^2 + 4,  x^5 + 2 x^3 - x^2 - 2]
```

```
-2 + 2 x - x^2 + x^3
```

■ Computing resultants

Theorem 1.

$$\text{res}_x(f,\ g)\ =\ (-1)^{m\,n}\ \text{res}_x(g,\ f)$$

Proof.

We need to count the number of exchanges in the Sylvester matrix. This is somewhat similar to the bubble sort. Start with the top row and move to the bottom. It requires $n+m-1$ swaps. We do this for the first n rows. So, the total number is $n\,(n+m-1)\ =\ n\,m\ +n\,(n-1)$. Each swap introduces (-1) factor. QED.

Theorem 2.

If $g(x) = c$ is a constant, then

$$\text{res}_x(f,\ c)\ =\ c^m$$

Proof.

The Sylvester matrix is of size m **x** m and consists of all zeros except the main diagonal with c on it (in each row). QED.

Theorem 3.

If $\deg(g) \geq \deg(f)$

$$g = f * q + r,\quad k\ =\ \deg(r)$$

then

$$\text{res}_x(f,\ g)\ =\ \text{LC}(f)^{n-k}\ \text{res}_x(f,\ r)$$

Proof. Since $g = f*q+r$ we replace all b_k coefficients of $g(x)$ with $r = g - f\,q$. The determinant won't change, because this operation corresponds to subtracting linear combinations of rows. Result of this subtraction will lead to to $(n-k)$ **x**$(n-k)$ zeros in the left lower corner of the Sylvester matrix.

$$S = \begin{pmatrix} R & * \\ 0 & S_1 \end{pmatrix}$$

Here R is a tringular matrix. As in the original Sylvester matrix S, the main diagonal of R consists of

LC(f).

Thus,

$$\det(S) = \begin{vmatrix} R & * \\ 0 & S_1 \end{vmatrix} = \mathrm{LC}(f)^{n-k} \det(S_1)$$

QED.

These three theorems immediately lead to the recursive algorithm for computing a resultant.

■ Example.

Compute the resultant of the following polynomials

$$f = 2x^5 - 3x^4 + 2x^3 - 4x^2 + 4;$$

$$g = x^5 + 2x^3 - x^2 - 3;$$

step 1. Divide g by f

```
PolynomialRemainder[x^5 + 2 x^3 - x^2 - 3, 2 x^5 - 3 x^4 + 2 x^3 - 4 x^2 + 4, x]
```

```
-5 + x^2 + x^3 + 3 x^4 / 2
```

$$g = \frac{3x^4}{2} + x^3 + x^2 - 5 \pmod f$$

Hence

$$\mathrm{res}_x(f,\ g) \ = \ 2^{5-4}\,\mathrm{res}_x\left(f,\ r_1 = \frac{3x^4}{2} + x^3 + x^2 - 5\right)$$

Applying theorem 1

$$\mathrm{res}_x(f,\ g) \ = \ 2\,\mathrm{res}_x(f,\ r_1) = \ 2\,(-1)^{5*4}\,\mathrm{res}_x(r_1,\ f) = \ 2\ \mathrm{res}_x(r_1,\ f)$$

step 2. Divide f by r_1

$$f = \frac{32x^3}{9} - \frac{10x^2}{9} + \frac{20x}{3} - \frac{94}{9} \pmod{r_1}$$

Hence

$$\mathrm{res}_x(r_1,\ f) \ = \ \left(\frac{3}{2}\right)^{5-3}\ \mathrm{res}_x\left(r_1,\ r_2 = \frac{32x^3}{9} - \frac{10x^2}{9} + \frac{20x}{3} - \frac{94}{9}\right)$$

and

$$\mathrm{res}_x(f,\ g)\ = 2\left(\frac{3}{2}\right)^2 \mathrm{res}_x(r_1,\ r_2) = 2\left(\frac{3}{2}\right)^2 \mathrm{res}_x(r_2,\ r_1)$$

step 3. Divide r_1 by r_2

$$r_1 = -\frac{693\,x^2}{512} + \frac{423\,x}{256} - \frac{351}{512}\ (\mathrm{mod}\ r_2)$$

Hence

$$\mathrm{res}_x(r_2,\ r_1) = \left(\frac{32}{9}\right)^{4-2} \mathrm{res}_x\left(r_2,\ r_3 = -\frac{693\,x^2}{512} + \frac{423\,x}{256} - \frac{351}{512}\right)$$

and

$$\mathrm{res}_x(f,\ g)\ = 2\left(\frac{3}{2}\right)^2\left(\frac{32}{9}\right)^2 \mathrm{res}_x(r_2,\ r_3) = 2\left(\frac{3}{2}\right)^2\left(\frac{32}{9}\right)^2 \mathrm{res}_x(r_3,\ r_2)$$

step 4. Divide r_2 by r_3

$$r_2 = \frac{52\,224\,x}{5929} - \frac{644\,608}{53\,361}\ (\mathrm{mod}\ r_3)$$

Hence

$$\mathrm{res}_x(r_3,\ r_2) = \left(-\frac{693}{512}\right)^{3-1} \mathrm{res}_x\left(r_3,\ r_4 = \frac{52\,224\,x}{5929} - \frac{644\,608}{53\,361}\right)$$

and

$$\mathrm{res}_x(f,\ g)\ = 2\left(\frac{3}{2}\right)^2\left(\frac{32}{9}\right)^2\left(-\frac{693}{512}\right)^2 \mathrm{res}_x(r_4,\ r_3)$$

step 5. Divide r_3 by r_4

$$r_3 = -\frac{46\,275\,845}{47\,941\,632}\ (\mathrm{mod}\ r_4)$$

Hence

$$\mathrm{res}_x(r_4,\ r_3) = \left(\frac{52\,224}{5929}\right)^{2-0} \mathrm{res}_x\left(r_4,\ r_5 = -\frac{46\,275\,845}{47\,941\,632}\right)$$

and

$$\mathrm{res}_x(f,\ g)\ = 2\left(\frac{3}{2}\right)^2\left(\frac{32}{9}\right)^2\left(\frac{693}{512}\right)^2\left(\frac{52\,224}{5929}\right)^2 \mathrm{res}_x(r_4,\ r_5)$$

step 6. We terminate computation by applying theorem 2

$$\mathrm{res}_x\left(r_4,\ -\frac{46\,275\,845}{47\,941\,632}\right) = \left(-\frac{46\,275\,845}{47\,941\,632}\right)^{\deg(r_4)}$$

$$\mathrm{res}_x(f,\ g)\ = 2\left(\frac{3}{2}\right)^2\left(\frac{32}{9}\right)^2\left(\frac{693}{512}\right)^2\left(\frac{52\,224}{5929}\right)^2\left(-\frac{46\,275\,845}{47\,941\,632}\right) = -7805$$

References

E. Hermite, Sur l'intégration des fractions rationelles, *Nouvelles Annales de Mathématiques*, **11**(1872), 145-148.

M. W. Ostrogradsky, De l'intégration des fractions rationelles, *Bulletin de la Classe Physico-Mathématiques de l'Académie Impériale des Sciences de St. Pétersburg*, IV, 1845, pp145-167, 286-300.

M. Bronstein, *Symbolic Integration - Transcendental Functions,* Algorithms and Computations in Mathematics, Vol 1, Springer-Verlag, 1996.