Gröbner Bases

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Application to Trigonometry

Problem. Compute exact values for $\cos\left(\frac{\pi}{7}\right)$ and $\sin\left(\frac{\pi}{7}\right)$.

We start with

$$-1 = \exp^7\left(\frac{i\,\pi}{7}\right) = \left(\cos\left(\frac{\pi}{7}\right) + i\sin\left(\frac{\pi}{7}\right)\right)^7$$

and expand the rhs and then separate the real and imaginary parts.

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ComplexExpand[(Cos[Pi/7] + I Sin[Pi/7])^7]
```

$$\mathrm{Cos}\left[\frac{\pi}{7}\right]^7 - 21\,\mathrm{Cos}\left[\frac{\pi}{7}\right]^5 \mathrm{Sin}\left[\frac{\pi}{7}\right]^2 + 35\,\mathrm{Cos}\left[\frac{\pi}{7}\right]^3 \mathrm{Sin}\left[\frac{\pi}{7}\right]^4 - 7\,\mathrm{Cos}\left[\frac{\pi}{7}\right] \mathrm{Sin}\left[\frac{\pi}{7}\right]^6 + i\left(7\,\mathrm{Cos}\left[\frac{\pi}{7}\right]^6 \mathrm{Sin}\left[\frac{\pi}{7}\right] - 35\,\mathrm{Cos}\left[\frac{\pi}{7}\right]^4 \mathrm{Sin}\left[\frac{\pi}{7}\right]^3 + 21\,\mathrm{Cos}\left[\frac{\pi}{7}\right]^2 \mathrm{Sin}\left[\frac{\pi}{7}\right]^5 - \mathrm{Sin}\left[\frac{\pi}{7}\right]^7\right)$$

```
{re, im} = {1 + Re[%], Im[%]};
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{im, re} = {im, re} /. {Cos[π/7] → c, Sin[π/7] → s}
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$$\left\{7\,c^6 s - 35\,c^4 s^3 + 21\,c^2 s^5 - s^7,\ 1 + c^7 - 21\,c^5 s^2 + 35\,c^3 s^4 - 7\,c\,s^6\right\}$$

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GroebnerBasis[{im, re, c^2 + s^2 - 1}, {}, {s}]
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$$\left\{1 - 3\,c - 8\,c^2 + 4\,c^3 + 8\,c^4\right\}$$

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Factor[%[[1]]]
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(1 + c) (1 - 4 c - 4 c^2 + 8 c^3)
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```
Solve[1 - 4 c - 4 c^2 + 8 c^3 == 0, c]
```

We proved

$$\cos\left(\frac{\pi}{7}\right) = \frac{1}{6} + \frac{7^{2/3}}{3 \times 2^{2/3} \sqrt[3]{-1 + 3\, i \sqrt{3}}} + \frac{7^{1/3}}{3\ 2^{4/3}} \sqrt[3]{-1 + 3\, i \sqrt{3}}$$

How do you find an algebaraic form for $\sin\left(\frac{\pi}{7}\right)$?

Hunting the pentagon or areas of cyclic polygons

A polygon inscribed in a circle is called a **cyclic** polygon.

Main problem: how to express the area of a cyclic polygon in terms of sides?

For polygons of more than three sides, the length of its sides do not determine a polygon and its area. That is why we impose a condition that a polygon is inscribed in a circle.

$$A = \frac{1}{2} \begin{vmatrix} x_1, & x_2 \\ y_1, & y_2 \end{vmatrix} + \frac{1}{2} \begin{vmatrix} x_2, & x_3 \\ y_2, & y_3 \end{vmatrix} + \ldots + \frac{1}{2} \begin{vmatrix} x_n, & x_1 \\ y_n, & y_1 \end{vmatrix}$$

Each side of the polygon is computed by

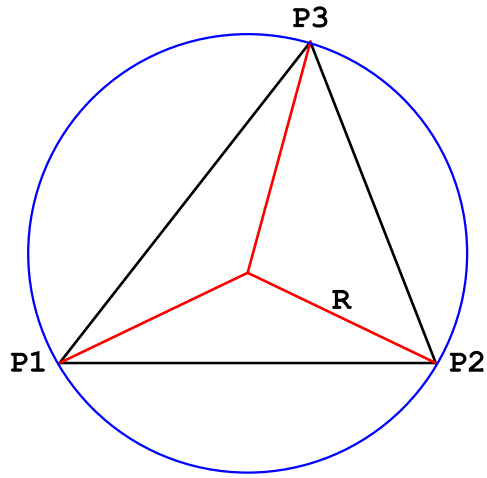
$$(x_k - x_{k+1})^2 + (y_k - y_{k+1})^2 = {s_k}^2, \quad k = 1, 2, \ldots, n$$
$$x_{n+1} = x_1,$$
$$y_{n+1} = y_1$$

Also we count distances from the origin (r_x, r_y) to each vertex

$$(x_k - r_x)^2 + (y_k - r_y)^2 = R^2, \quad k = 1, 2, \ldots, n$$

Altogether, there are $2\,n + 1$ equation and $2\,n + 4$ unknown variables. Without loss of generality, we assume that $x_1 = x_2 = y_1 = 0$ then we obtain the system of $2\,n + 1$ algebraically independent equations with $2\,n + 1$ unknowns.

■ **Heron's formula** (first century BC)




```
p1 = {0, 0}; p2 = {x2, 0}; p3 = {x3, y3}; Cr = {rx, ry};
GroebnerBasis[{
  A - 1/2 Det[{p1, p2}] - 1/2 Det[{p2, p3}] - 1/2 Det[{p3, p1}],
  s1^2 - (p2 - p1) . (p2 - p1),
  s2^2 - (p3 - p2) . (p3 - p2),
  s3^2 - (p3 - p1) . (p3 - p1),
  R^2 - (Cr - p1) . (Cr - p1),
  R^2 - (Cr - p2) . (Cr - p2),
  R^2 - (Cr - p3) . (Cr - p3)
 }, {R, s1, s2, s3}, {rx, ry, x2, x3, y3}]
```

The first equation is known as Heron's fomula. that is usually written as

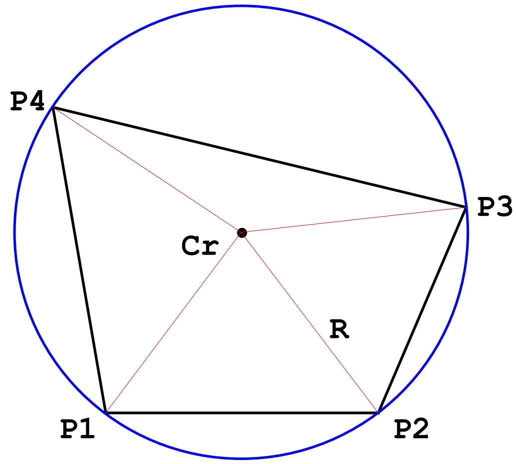
$$A = \sqrt{p\,(p - s_1)\,(p - s_2)\,(p - s_3)}$$

p is a half-perimeter.

The second equation unites the area with the radius

$$4\,A\,R = s_1\,s_2\,s_3$$

■ **Brahmagupta's formula** (seventh century)




```
p1 = {0, 0}; p2 = {x2, 0}; p3 = {x3, y3}; p4 = {x4, y4}; Cr = {rx, ry};
gb = GroebnerBasis[{
    A - 1/2 Det[{p1, p2}] - 1/2 Det[{p2, p3}] - 1/2 Det[{p3, p4}] - 1/2 Det[{p4, p1}],
    s1^2 - (p2 - p1) . (p2 - p1),
    s2^2 - (p3 - p2) . (p3 - p2),
    s3^2 - (p4 - p3) . (p4 - p3),
    s4^2 - (p4 - p1) . (p4 - p1),
    R^2 - (Cr - p1) . (Cr - p1),
    R^2 - (Cr - p2) . (Cr - p2),
    R^2 - (Cr - p3) . (Cr - p3),
    R^2 - (Cr - p4) . (Cr - p4)
   }, {},
   {x2, x3, y3, x4, y4, rx, ry, R},
   MonomialOrder → EliminationOrder]
```

We derived the Brahmagupta formula: which is usually written in the following symmetric form

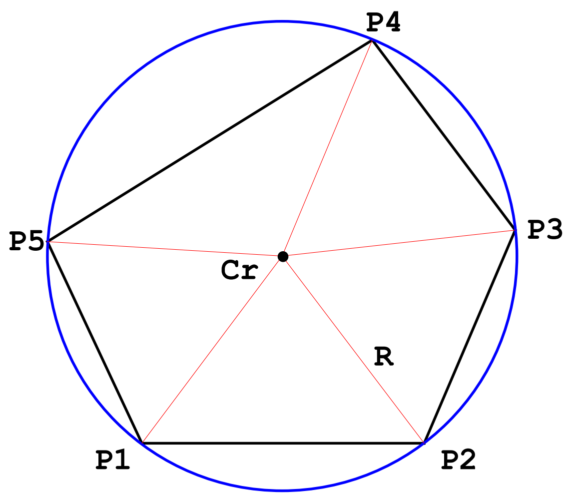
$$16\,A^2 = (-\mathrm{s1} + \mathrm{s2} + \mathrm{s3} + \mathrm{s4})\,(\mathrm{s1} - \mathrm{s2} + \mathrm{s3} + \mathrm{s4})\,(\mathrm{s1} + \mathrm{s2} - \mathrm{s3} + \mathrm{s4})\,(\mathrm{s1} + \mathrm{s2} + \mathrm{s3} - \mathrm{s4})$$

In similar way we obtain a relation between area and radius:

$$(4\,A\,R)^2 = (\mathrm{s2}\,\mathrm{s3} + \mathrm{s1}\,\mathrm{s4})\,(\mathrm{s1}\,\mathrm{s3} + \mathrm{s2}\,\mathrm{s4})\,(\mathrm{s1}\,\mathrm{s2} + \mathrm{s3}\,\mathrm{s4})$$

■ **Robbins' formula** (1995)

It may be surprising that so long time has elapsed... We will see that calculation leading to Robbons's formula have signficant computational abstacles.




```
p1 = {0, 0}; p2 = {x2, 0}; p3 = {x3, y3}; p4 = {x4, y4}; p5 = {x5, y5};
Cr = {rx, ry};
gb = GroebnerBasis[{
    A - 1/2 Det[{p1, p2}] - 1/2 Det[{p2, p3}] - 1/2 Det[{p3, p4}] -
     1/2 Det[{p4, p5}] - 1/2 Det[{p5, p1}],
    s1^2 - (p2 - p1) . (p2 - p1),
    s2^2 - (p3 - p2) . (p3 - p2),
    s3^2 - (p4 - p3) . (p4 - p3),
    s4^2 - (p4 - p5) . (p4 - p5),
    s5^2 - (p5 - p1) . (p5 - p1),
    R^2 - (Cr - p1) . (Cr - p1),
    R^2 - (Cr - p2) . (Cr - p2),
    R^2 - (Cr - p3) . (Cr - p3),
    R^2 - (Cr - p4) . (Cr - p4),
    R^2 - (Cr - p5) . (Cr - p5)
   }, {},
   {x2, x3, y3, x4, y4, x5, y5, rx, ry, R},
   MonomialOrder → EliminationOrder];
```

This approach leads to a single polynomial equation for the area in terms of sides. The order of equation is 28 and it takes about 10Mb of space.

■ Robbins's work

The Wall Street Journal (2003):

If you had just a short time to live, what would you do with it? David Robbins, a mathematician at the Center for Communications Research, Princeton, was solving a really tough geometry problem.

available at *http://www.plambeck.org/oldhtml/mathematics/robbins.htm*

In his work, Robbins computed cases $n = 5$ and $n = 6$.

D. P. Robbins, Areas of polygons inscribed in a circle, *Discrete Comput. Geom.*, 12(1994), 223-236.

D. P. Robbins, Areas of polygons inscribed in a circle, *Amer. Math. Monthly*, 102(1995), 523-530.

Robbins was not able to solve the system above directly. Instead, he gained some insights by running a numerical experiment and then guessed the formula. He used *Mathematica*.

The order of a polynomial for the area has as many real roots as the number of different areas of self-intersecting polygons. He chose numbers 29,30,31,32,33 and draw all cyclic pentagons of these lengths. Then he constructed a polynomial with these seven roots. The next his guess was on exact forms of coefficients - they must be terms of symmetric functions. At the end he had a system of 70 linear equations that was easy to solve.

Robbins conjected the general form of the relation between an area and sides for any n.

Robbins' conjecture on the order of a minimal polynomial

$$A(2k+1) = \frac{1}{2}(2k+1)\binom{2k}{k} - 2^{2k-1}$$

$$A(2k+2) = (2k+1)\binom{2k}{k} - 2^{2k-1}$$

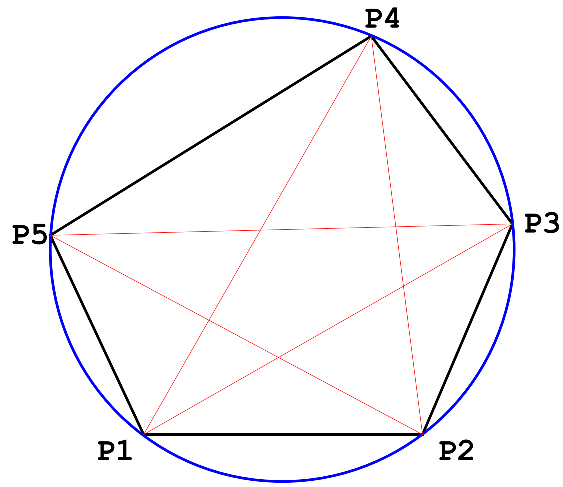
The conjecture was recently proved

M. Fedorchuk and I. Pak, Rigidity and polynomial invariants of convex polytopes, *Duke Math. J.* (2005); available at

http://www-math.mit.edu/~pak/research.html

■ **The Gauss pentagon formula** (1823)

Seems nobody was aware of this result until Prof. Wu revived this formula in his book on *Mathematics Mechanization*, p.327. The formula says that to compute the area we need to go around the pentagon and measure areas of its vertex triangles (that which formed by three conseciutive vertices).



Given a cyclic pentagon. If b_0, b_1, b_2, b_3, b_4 *are areas of the vertex triangles then the pentagon area is given by*

$$A^2 - A(b_0 + b_1 + b_2 + b_3 + b_4) + b_0\, b_1 + b_1\, b_2 + b_2\, b_3 + b_3\, b_4 + b_4\, b_0$$

References

[1] Wen-tsun Wu, *Mathematics Mechanization*, Kluwer Acad. Publ., Beijing, 2000.

[2] D. P. Robbins, Areas of polygons inscribed in a circle, *Discrete Comput. Geom*., 12(1994), 223-236.

[3] D. P. Robbins, Areas of polygons inscribed in a circle, *Amer. Math. Monthly*, **102**(1995), 523-530.

[4] M. Fedorchuk and I. Pak, Rigidity and polynomial invariants of convex polytopes, *Duke Math. J.,* **129**(2005), 371-404.

[5] M. Trott, *The Mathematica GuideBooks*, Springer, New York, 2005.