Symbolic Summation

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Gosper's Algorithm

"The ultimate goal of mathematics is to eliminate

all need for intelligent thought"

D. Knuth [2, p. 56]

Introduction

Consider

$$S(n) = \sum_{k=1}^{n} k\, k!$$

It satisfies to the following recurrence equation for S

$$S(n) - S(n-1) = n\, n!$$

How to solve it? Using the method of iteration will lead back to the original sum. Fortunately, we can use a simple relation for a factorial function:

$$(n+1)! = (n+1)\, n! = n\, n! + n!$$

It follows

$$n\, n! = (n+1)! - n!$$

Therefore,

$$\sum_{k=1}^{n} k\, k! = \sum_{k=1}^{n} ((k+1)! - k!) =$$

$$(2! - 1!) + (3! - 2!) + (4! - 3!) + \ldots ((n+1)! - n!) = (n+1)! - 1$$

and we can find a closed form for that sum without solving any recurrence equations.

■ Main Idea

Given

$$\sum_{k=1}^{n} s_k$$

we want to find a new sequence z_k such that

$$s_k = z_{k+1} - z_k \tag{1}$$

If we succeed in finding z_k then the definite sum can be computed by telescoping

$$\sum_{k=1}^{n} s_k = \sum_{k=1}^{n} (z_{k+1} - z_k) = z_{n+1} - z_1$$

Thus, the problem of summation is reduced to finding sequence z_k. Let us consider the ratio $\frac{z_k}{s_k}$ and assume that is a rational function:

$$\frac{z_k}{s_k} = \frac{z_k}{z_{k+1} - z_k} = \frac{1}{\frac{z_{k+1}}{z_k} - 1} \in Q(k) \tag{2}$$

This is so if z_k is a hypergeometric term. Since (2), we can write

$$z_k = y_k\, s_k$$

where $y_k \in Q(k)$ is a unknown rational function. Substituting it back into (1)

$$s_k = z_{k+1} - z_k = y_{k+1}\, s_{k+1} - y_k\, s_k$$

and dividing it by s_k

$$1 = y_{k+1} \frac{s_{k+1}}{s_k} - y_k$$

we obtain a recurrence equation for y_k, where $\frac{s_{k+1}}{s_k} \in Q(k)$. The latter must be proved. Indeed,

$$\frac{s_{k+1}}{s_k} = \frac{z_{k+2} - z_{k+1}}{z_{k+1} - z_k} = \frac{z_{k+1}}{z_k} \frac{\frac{z_{k+2}}{z_{k+1}} - 1}{\frac{z_{k+1}}{z_k} - 1} \in Q(k)$$

since z_k is a hypergeometric term. Therefore, we reduce summation problem to finding *rational solutions* to

$$a_k\, y_{k+1} - y_k = 1$$

where $a_k = \frac{s_{k+1}}{s_k}$ is some known rational function. How would you find a rational solution? Wouldn't it be much easier to find a polynomial solution?

In the next step, we will show that we can simplify the problem further to finding only polynomial solutions. Assume (this will be proven later-!), we can rewrite any rational function in the following special form:

$$a_k = \frac{s_{k+1}}{s_k} = \frac{p_{k+1}\, q_{k+1}}{p_k\, r_{k+1}} \tag{3}$$

where p, q and r are polynomials and

$$\text{GCD}(q_k,\ r_{k+j}) = 1, \quad j \in N_0.$$

Next, we define f_k such that

$$f_k = \frac{p_{k+1}}{r_{k+1}} \frac{z_{k+1}}{s_{k+1}}$$

Clearly f_k is rational

$$f_k = \frac{p_{k+1}}{r_{k+1}} \frac{z_{k+1}}{s_{k+1}} = \frac{p_{k+1}}{r_{k+1}} \frac{z_{k+1}}{z_{k+2} - z_{k+1}} = \frac{p_{k+1}}{r_{k+1}} \frac{1}{\frac{z_{k+2}}{z_{k+1}} - 1} \in Q(k)$$

But we can say more about f_k - it is actually a polynomial. Again, this will be proven later. Let us continue with the main idea. Using f_k we can rewrite sequence z_k from (1) as

$$z_k = \frac{r_k}{p_k}\, s_k\, f_{k-1} \tag{4}$$

and therefore,

$$s_k = z_{k+1} - z_k = \frac{r_{k+1}}{p_{k+1}}\, s_{k+1}\, f_k - \frac{r_k}{p_k}\, s_k\, f_{k-1}$$

Dividing it by s_k

$$1 = \frac{r_{k+1}}{p_{k+1}} \frac{s_{k+1}}{s_k}\, f_k - \frac{r_k}{p_k}\, f_{k-1}$$

multiplying it by p_k

$$p_k = \frac{p_k}{p_{k+1}} \frac{s_{k+1}}{s_k}\, r_{k+1}\, f_k - r_k\, f_{k-1}$$

and using (3)

$$\frac{s_{k+1}}{s_k} = \frac{p_{k+1}\, q_{k+1}}{p_k\, r_{k+1}} \Longrightarrow \frac{p_k}{p_{k+1}} \frac{s_{k+1}}{s_k} = \frac{q_{k+1}}{r_{k+1}}$$

arrive at the following recurrence equation with polynomial coefficients for f_k

$$p_k = q_{k+1}\ f_k - r_k\ f_{k-1} \tag{5}$$

Rememeber, that f_k is a polynomial! So, we are interested only in polynomial solutions to (5). This can be easily done by using a generic polynomial (what degree-?) with unknown coefficients, substituting it into (5), equating all coefficients by k to zero to get a system of linear equations from which we could find coefficents of the original generic polynomial. Once we have f_k, we get z_k

$$z_k = \frac{r_k}{p_k}\ s_k\ f_{k-1}$$

and finally using (1), we find the result of summation.

$$\sum_{k=1}^{n} s_k = \sum_{k=1}^{n} (z_{k+1}\ -\ z_k) = z_{n+1} - z_1$$

■ **We need to prove**

1. representation (3) always exists
2. f_k is a polynomial
3. a priori bound on the degree of f_k

Observe, the algorithm is invariant to the top limit of summation. Therefore, we say that Gosper's algorithm is designed for **indefinite summation**

$$\sum_k s_k = z_k$$

We define indefinite summation in the following way: given a sequence s_k, find another sequence z_k such that

$$s_k = z_{k+1} - z_k = \Delta\ z_k$$

Note, indefinite summation in the inverse of Δ. Similar, as integration is the inverse of differentiation.

Knowing sequence z_k it is easy to compute a definite sum:

$$\sum_{k=m}^{n} s_k = \sum_{k=m}^{n} \Delta\ z_k = z_{n+1} - z_m$$

■ Algorithm for undefinite summation

To sum

$$\sum_{k=1}^{n} s_k \text{ or } \sum_{k} s_k$$

1. Compute

$$\frac{s_{k+1}}{s_k}$$

and verify that it is hypergeometric.

2. Find polynomials p, q, and r such that

$$\frac{s_{k+1}}{s_k} = \frac{p_{k+1}}{p_k} \frac{q_{k+1}}{r_{k+1}} \quad \text{where } \mathrm{GCD}(q_k, \ r_{k+j}) = 1, \quad j \in N_0.$$

3. Find a polynomial solution to

$$p_k = q_{k+1} \ f_k - r_k \ f_{k-1}$$

4. If such solution exists, return

$$z_k = \frac{r_k}{p_k} \ s_k \ f_{k-1}$$

otherwise terminate the algotithm.

■ Example 1.

$$\sum_{k=0}^{n} \frac{(-1)^k}{1+k} \binom{n}{k}$$

1. Compute

$$\frac{s_{k+1}}{s_k} = \frac{k-n}{2+k}$$

2. Choose p, q, and r, such that

$$\frac{s_{k+1}}{s_k} = \frac{p_{k+1}}{p_k} \frac{q_{k+1}}{r_{k+1}}, \quad \text{where } \mathrm{GCD}(q_k, \ r_{k+j}) = 1, \quad j \in N_0.$$

Obviously,

$p_k = 1;$
$q_{k+1} = k - n,$

$$r_{k+1} = 2 + k$$

3. Find a polynomial solution to $p_k = q_{k+1}\, f_k - r_k\, f_{k-1}$ which is

$$1 = (k - n)\, f_k - (k + 1)\, f_{k-1}$$

The polynomial solution is a constant $f_k = c$

$$1 = (k - n)\, c - (k + 1)\, c$$

$$1 = -n\, c - c$$

$$c = -\frac{1}{n + 1}$$

4. Return $z_k = \frac{r_k}{p_k}\, s_k\, f_{k-1}$ which is

$$z_k = \frac{r_k}{p_k}\, s_k\, f_{k-1} = \frac{1 + k}{1} * \frac{(-1)^k}{1 + k} \binom{n}{k} * \frac{(-1)}{n + 1} = \binom{n}{k} \frac{(-1)^{k+1}}{n + 1}$$

Therefore,

$$\sum_{k=0}^{n} s_k = z_{n+1} - z_0 = \binom{n}{n + 1} \frac{(-1)^{n+2}}{n + 1} - \binom{n}{0} \frac{(-1)}{n + 1} = \frac{1}{n + 1}$$

■ Example 2.

If we generalize the denominator a little

$$\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{k + z}$$

the sum is not computable by Gosper's algorithm. Prove it! Though, there is a closed form solution for a <u>definite</u> sum

$$\sum_{k=0}^{n} \frac{(-1)^k}{k + z} \binom{n}{k} = \frac{1}{z \binom{n + z}{z}}$$

■ *Mathematica* session

In this section we provide all steps of Gosper's algorithm the way they can be done in *Mathematica*. As an example, we evaluate

$$\sum_k \frac{(-1)^k}{\binom{n}{k}}$$

First we define the summand as a *Mathematica* function

```
s[k_] := (-1)^k / Binomial[n, k]
```

Then we compute the ratio

```
s[k + 1] / s[k] // FunctionExpand // Simplify
```

```
(1 + k) / (k - n)
```

Since it's a hypergeometric term, we find a triple $(p_k,\ q_k,\ r_k)$ such that

$$\frac{s_{k+1}}{s_k} = \frac{p_{k+1}}{p_k}\ \frac{q_{k+1}}{r_{k+1}}$$

```
p[k_] := 1
q[k_] := k
r[k_] := k - 1 - n
```

Verify the polynomial GCD (see the algorithm for that down below)

```
Table[PolynomialGCD[q[k], r[k + j]], {j, 0, 15}]
```

```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Here is a difference equation for f_k

$$p_k = q_{k+1}\ f_k - r_k\ f_{k-1}$$

```
p[k] == q[k + 1] f[k] - r[k] f[k - 1]
```

```
1 == - (-1 + k - n) f[-1 + k] + (1 + k) f[k]
```

The solution is obvious

```
Collect[- (-1 + k - n) f[-1 + k] + (1 + k) f[k] - 1 /. f[k_] :→ a, k,
 Factor]
```

```
-1 + 2 a + a n
```

Solving it, we find a

```
Solve[% == 0, a]
```

$$\left\{\left\{a \to \frac{1}{2+n}\right\}\right\}$$

Thus,

$$z_k = \frac{r_k}{p_k} f_{k-1} s_k$$

and

$$\sum_k \frac{(-1)^k}{\binom{n}{k}} = \frac{(-1)^k (k-n-1)}{(n+2)\binom{n}{k}}$$

In particular,

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} = \frac{(1+(-1)^n)(n+1)}{n+2}$$

■ **GCD(q_k, r_{k+j}) = 1, $j \in N_0$.**

If they are have different degrees, we factor them over rationals and apply the procedure to each pair of factors. Therefore, we assume that q and r are of the same degree. Let

$$q_k = a k^n + b k^{n-1} + \ldots$$

$$r_k = c k^n + d k^{n-1} + \ldots$$

Consider

$$\frac{c}{a} q_k = c k^n + \frac{b c}{a} k^{n-1} + \ldots \tag{6}$$

From the GCD$\left(q_k, r_{k+j}\right) \neq 1$

$$\frac{c}{a}\, q_k = r_{k+j} \tag{7}$$

Evaluating the right hand side, we obtain

$$r_{k+j} = c\,(k+j)^n + d\,(k+j)^{n-1} + \ldots = c\,k^n + n\,c\,j\,k^{n-1} + d\,k^{n-1} + \ldots$$

Next, we put this and (6), back to equation (7)

$$c\,k^n + \frac{b\,c}{a}\,k^{n-1} + \ldots = c\,k^n + n\,c\,j\,k^{n-1} + d\,k^{n-1} + \ldots$$

Comparing coefficients by k^{n-1}, yields the value for j

$$\frac{b\,c}{a} = n\,c\,j + d$$

$$j = \frac{b\,c - a\,d}{a\,c\,n}$$

Note, j must be non-negative integer in order for the GCD not to be one. Finally, we have to verify (why-?)

$$c\,q_k - a\,r_{k+j} = 0$$

Example.

```
q[k_] := 5 k^2 + 2 k + 5
r[k_] := 5 k^2 - 48 k + 120
n := Exponent[q[k], k]
a := Coefficient[q[k], k, n]
b := Coefficient[q[k], k, n - 1]
c := Coefficient[r[k], k, n]
d := Coefficient[r[k], k, n - 1]
j = (b c - a d)/(a c n)
```

```
5
```

```
c q[k] - a r[k + j] // Expand
```

```
0
```

Gosper's Algorithm (proofs)

"In the first place, the beginner must be convinced that proofs deserve to be studied, that they have a purpose, that they are interesting" -

George Polya

■ We need to prove.

1. for any rational function

$$\frac{s_{k+1}}{s_k} = \frac{p_{k+1}}{p_k}\,\frac{q_{k+1}}{r_{k+1}}$$

where p, q and r are polynomials. and

$$\mathrm{GCD}(q_k,\; r_{k+j}) = 1, \quad j \in N_0.$$

2. f_k is a polynomial

$$f_k = \frac{p_{k+1}}{r_{k+1}}\,\frac{z_{k+1}}{s_{k+1}}$$

3. a priori bound on the degree of f_k

$$p_k = q_{k+1}\,f_k - r_k\,f_{k-1}$$

■ Proof of 1.

Let us choose

$$p_k = 1, \quad q_k = s_k, \quad \text{and } r_k = s_{k-1}$$

$$\frac{s_{k+1}}{s_k} = \frac{p_{k+1}}{p_k}\,\frac{q_{k+1}}{r_{k+1}}$$

If

$$\mathrm{GCD}(q_k,\; r_{k+j}) = 1, \quad j \in N_0.$$

there is nothing to prove. Assume that

$$\mathrm{GCD}(q_k,\; r_{k+j}) = \pi_k \neq 1, \quad \text{for some } j \in J.$$

We choose new functions P_k, Q_k, and R_k such that

$$\frac{P_{k+1}}{P_k}\,\frac{Q_{k+1}}{R_{k+1}} = \frac{p_{k+1}}{p_k}\,\frac{q_{k+1}}{r_{k+1}}$$

They are

$$Q_k = \frac{q_k}{\pi_k}, \quad R_k = \frac{r_k}{\pi_{k-j}} \text{ and } P_k = p_k\,\pi_k\,\pi_{k-1}\ldots\pi_{k-j+1},$$

Simple check

$$\frac{P_{k+1}}{P_k}\,\frac{Q_{k+1}}{R_{k+1}} = \frac{p_{k+1}\,\pi_{k+1}\,\pi_k\ldots\pi_{k-j+2}}{p_k\,\pi_k\,\pi_{k-1}\ldots\pi_{k-j+2}\,\pi_{k-j+1}}\,\frac{q_{k+1}}{\pi_{k+1}}\,\frac{\pi_{k-j+1}}{r_{k+1}} = \frac{p_{k+1}}{p_k}\,\frac{q_{k+1}}{r_{k+1}}$$

It follows then

$$\text{GCD}(Q_k,\ R_{k+j}) = \text{GCD}\left(\frac{q_k}{\pi_k},\ \frac{r_{k+j}}{\pi_k}\right) = 1$$

Doing this for all $j \in J$ we find triples $(p_k,\ q_k,\ r_k)$ we conclude the proof.■

■ **Proof of 2.**

The proof is by contradiction to $\text{GCD}(q_k,\ r_{k+j}) = 1, \quad j \in N_0$.. Assume that f_k is rational

$$f_k = \frac{a_k}{b_k}, \quad \gcd(a_k,\ b_k) = 1$$

where b_k is NOT a constant. Substitute this f_k into the equation

$$p_k = q_{k+1}\,f_k - r_k\,f_{k-1}$$

we obtain

$$p_k = q_{k+1}\,\frac{a_k}{b_k} - r_k\,\frac{a_{k-1}}{b_{k-1}}$$

or (after multiplying by $b_k\,b_{k-1}$)

$$b_k\,b_{k-1}\,p_k = b_{k-1}\,q_{k+1}\,a_k - b_k\,r_k\,a_{k-1} \tag{8}$$

Suppose N is the largest integer such that (why the largest N exists -?)

$$\gcd(b_k,\ b_{k+N}) = g_k \neq 1 \tag{9}$$

It follows from (9) that g_k divides b_k. Therefore, g_k divides $b_{k-1}\,q_{k+1}\,a_k$.

g_k does not divide a_k, because $\gcd(a_k,\ b_k) = 1$

g_k does not divide $b_{k-1}$by (9)

thus g_k divides q_{k+1} which is the same as g_{k-1} divides q_k

It follows from (9) that g_{k-N-1} divides b_{k-1} (by performing simple shifting $k \to k - N - 1$). Therefore, in view of (6) g_{k-N-1} divides $b_k\, r_k\, a_{k-1}$

g_{k-N-1} does not divide b_k by (9) (N is the maximum)

g_{k-N-1} does not divide a_{k-1}, because otherwise g_k would divide a_{k+N} and then $\gcd(a_{k+N},\ b_{k+N}) \neq 1$

thus g_{k-N-1} divides r_k which is the same as g_{k-1} divides r_{k+N}

This contradicts to

$$\mathrm{GCD}\left(q_k,\ r_{k+j}\right) = 1, \quad j \in N_0.$$

for $j = N$.QED

■ Proof of 3.

Here we find the upper bound on the degree of f_k

$$p_k = q_{k+1}\, f_k - r_k\, f_{k-1}$$

The equation can be rewritten as

$$p_k = (q_{k+1} - r_k)\, \frac{f_k + f_{k-1}}{2} + (q_{k+1} + r_k)\, \frac{f_k - f_{k-1}}{2} \tag{10}$$

It is easy to see that

$$\deg\left[\frac{f_k + f_{k-1}}{2}\right] > \deg\left[\frac{f_k - f_{k-1}}{2}\right]$$

because the highest degree term will be canceled in $f_k - f_{k-1}$.

Which one of two terms in the rhs of (10) has a higher degree?

Case 1

If

$$\deg(q_{k+1} - r_k) \geq \deg(q_{k+1} + r_k)$$

then

$$\deg(f_k) = \deg(p_k) - \deg(q_{k+1} - r_k)$$

Proof .

Since

$$\deg\left[(q_{k+1} - r_k)\, \frac{f_k + f_{k-1}}{2}\right] > \deg\left[(q_{k+1} + r_k)\, \frac{f_k - f_{k-1}}{2}\right]$$

the second term in the rhd of (10) can be neglected

$$p_k = (q_{k+1} - r_k)\,\frac{f_k + f_{k-1}}{2}$$

Therefore,

$$\deg(p_k) = \deg(q_{k+1} - r_k) + \deg\left(\frac{f_k + f_{k-1}}{2}\right) = \deg(q_{k+1} - r_k) + \deg(f_k)$$

This concludes the first case.

Case 2

If $n = \deg(q_{k+1} + r_k) > \deg(q_{k+1} - r_k)$

then

a) if $\frac{2\,b}{a}$ is not negative integer

$$\deg(f_k) = \deg(p_k) - n + 1$$

b) if $\frac{2\,b}{a}$ is negative integer

$$\deg(f_k) \le \max\left[-2 * \frac{b}{a},\ \deg(p_k) - n + 1\right]$$

Proof Let

$$f_k = c * k^m + \ldots$$

$$q_{k+1} + r_k = a * k^n + \ldots, \quad \text{where } n = \deg(q_{k+1} + r_k)$$

$$q_{k+1} - r_k = b * k^{n-1} + \ldots \qquad \text{why } (n-1)?$$

Then by (10) we have

$$p_k = \left(b * k^{n-1} + \ldots\right)\frac{2 * c * k^m + \ldots}{2} + (a * k^n + \ldots)\,\frac{c * m * k^{m-1} + \ldots}{2}$$

Collecting terms

$$p_k = \left(b + \frac{a * m}{2}\right) * c * k^{n+m-1} + \ldots$$

Depending on whether or not $\frac{2\,b}{a}$ is negative, we have two choices. QED

Not Gosper-summable

$$\sum_{k=0}^{m} \binom{n}{k}$$

Compute the ratio

$$\frac{s_{k+1}}{s_k} = \frac{n-k}{k+1}$$

Obviously

$$p_k = 1;$$
$$q_{k+1} = n-k,$$
$$r_{k+1} = k+1$$

Compute the degree of f_k

$$\deg(q_{k+1} - r_k) = \deg(n-k-k) = 1$$
$$\deg(q_{k+1} + r_k) = \deg(n-k+k) = 0$$

This is the first case, therefore

$$\deg(f_k) = \deg(p_k) - \deg(q_{k+1} - r_k)$$
$$\deg(f_k) = -1$$

but f_k must be a polynomial, thus, the above sum is not Gosper-summable.

References

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