Computer Science 355
Modern Computer Algebra

Assignment 2
solutions

Problem 1 (Programming) (20 pts)

Write a program which takes an arbitrary list of integers as input, and produces a new list, which consists of {value, run-length} pairs describing the given input string. This is known as Run-Length Encoding. For example, the imput string

{ 7, 8, 8, 9, 9, 9, 9, 5, 5, 5, 5, 9, 9, 9 }

should produce the following output

{{ 7, 1 }, { 8, 2 }, { 9, 4 }, { 5, 4 }, { 9, 3 }}

■

```
RunLength[lst_List] :=
```

Problem 2

Evaluate the sum using Gosper's algorithm

$$\sum_k k \frac{\left(k - \frac{1}{2}\right)!^2}{(k+1)!^2}$$

Demonstrate each step of the algorithm.

$$s[k\_] := k \frac{\left(k - \frac{1}{2}\right)!^2}{(k+1)!^2}$$

$$\frac{s[k+1]}{s[k]} \text{ // FunctionExpand // Simplify}$$

$$\frac{(1+k)\ (1+2\ k)^2}{4\ k\ (2+k)^2}$$

```
p[k_] := k
q[k_] := (-1 + 2 k)^2
r[k_] := 4 (k + 1)^2
```

```
Table[PolynomialGCD[q[k], r[k + j]], {j, 0, 15}]
```

```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Note that all we did is check a bunch of cases, this isn't actually a proof that q and r are relatively prime the way we need them. To actually prove the relative primality, we would note that if there was a common factor, there would be a common root. However, q[k] has a double root at $\frac{1}{2}$while r[k+j] has roots at -(j+1) for all nonnegative integers j. These are all integers so there are no common roots.

```
q[k + 1] f[k] - r[k] f[k - 1] - p[k] /. f[k_] → a k + b
```

$$-k - 4 (b + a (-1 + k)) (1 + k)^2 + (b + a k) (-1 + 2 (1 + k))^2$$

```
Collect[%, k]
```

$$4 a - 3 b + (-1 + 5 a - 4 b) k$$

```
CoefficientList[%, k]
```

$$\{4 a - 3 b, -1 + 5 a - 4 b\}$$

```
Map[Equal[#, 0] &, %]
```

$$\{4 a - 3 b == 0, -1 + 5 a - 4 b == 0\}$$

```
Solve[%, {a, b}]
```

$$\{\{a \to -3, b \to -4\}\}$$

We now know that f[k]. Let's plug it in to get z.

$$\texttt{f[k\_] := -3 k - 4}$$

$$\texttt{z[k\_]} := \frac{\texttt{r[k]}}{\texttt{p[k]}} \texttt{s[k] f[k - 1]}$$

We can now do the indefinite summation.

$$\sum_k k \frac{\left(k - \frac{1}{2}\right)!^2}{(k+1)!^2} = -\frac{4 (1 + 3 k) \text{Gamma}\left[\frac{1}{2} + k\right]^2}{\text{Gamma}[1 + k]^2}$$

$$\texttt{Sum}\left[k \frac{\left(k - \frac{1}{2}\right)!^2}{(k+1)!^2}, k\right] \texttt{ // FullSimplify}$$

$$-\frac{4 (1 + 3 k) \text{Gamma}\left[\frac{1}{2} + k\right]^2}{\text{Gamma}[1 + k]^2}$$

Problem 3

Prove that this sum is not Gosper-summable

$$\sum_k (-4)^k \binom{n+k}{2\,k}$$

```
s[k_] := (-4)^k Binomial[n + k, 2 k]
```

```
s[k + 1]/s[k] // FunctionExpand // Simplify
```

$$\frac{2\,(k-n)\,(1+k+n)}{(1+k)\,(1+2\,k)}$$

```
p[k_] := 1
q[k_] := 2 (k + n) (k - n - 1)
r[k_] := k (2 k - 1)
```

```
Table[PolynomialGCD[q[k], r[k + j]], {j, 0, 15}]
```

{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

```
p[k + 1]/p[k] q[k + 1]/r[k + 1] // FullSimplify
```

$$\frac{2\,(k-n)\,(1+k+n)}{(1+k)\,(1+2\,k)}$$

```
q[k + 1] f[k] - r[k] f[k - 1] == p[k]
```

$-k\,(-1+2\,k)\,f[-1+k] + 2\,(k-n)\,(1+k+n)\,f[k] == 1$

We will prove that this equation has no polynomial solution.

```
Exponent[q[k + 1] + r[k], k] > Exponent[q[k + 1] - r[k], k]
```

True

It falls into Case 2.

```
nn = Exponent[q[k + 1] + r[k], k];
a = Coefficient[q[k + 1] + r[k], k, nn];
b = Coefficient[q[k + 1] - r[k], k, nn - 1];
```

```
2 b/a
```

$\frac{3}{2}$

It falls into Case 2 a)

```
Exponent[p[k], k] - nn + 1
```

-1

A polynomial of degree -1 does not exist. This proves that the sum is not Gosper-summable.

Problem 4

Let m be a positive integer, $p(k)$ a polynomial of degree $m-1$ and c a constant. Determine is the following sum is Gosper-summable

$$\sum_k \frac{p(k)}{\prod_{j=0}^{m-1}(k+c+j)}$$

■

$$\frac{s[k+1]}{s[k]} = \frac{p\,(k+1)}{p\,(k)}\ \frac{(k+c)}{(k+m+c)}$$

```
Clear[p, k, c, f];
q[k_] := k + c - 1
r[k_] := k + m + c - 1
```

```
q[k + 1] - r[k]
```

```
1 - m
```

```
q[k + 1] + r[k]
```

```
-1 + 2 c + 2 k + m
```

It falls into Case 2):

```
n = 1; a = 2; b = 1 - m;
```

$$\texttt{max}\left[-2\,\frac{b}{a},\ (m-1)\ -n\ +\ 1\right]$$

```
max[-1 + m, -1 + m]
```

so, degree of f is $m-1$

$$\texttt{q[k+1] f[k] - r[k] f[k-1] /. }\{\texttt{f[k\_]} :\!\to \texttt{d[m-1]}\,k^{m-1} + \texttt{d[m-2]}\,k^{m-2}\}$$

$$-(-1+c+k+m)\left((-1+k)^{-2+m}\,d[-2+m] + (-1+k)^{-1+m}\,d[-1+m]\right) + (c+k)\left(k^{-2+m}\,d[-2+m] + k^{-1+m}\,d[-1+m]\right)$$

$$\texttt{\% /. }(k-1)^{m-1} \to k^{m-1} - (m-1)\,k^{m-2}$$

$$(c+k)\left(k^{-2+m}\,d[-2+m] + k^{-1+m}\,d[-1+m]\right) - (-1+c+k+m)\left((-1+k)^{-2+m}\,d[-2+m] + \left(k^{-1+m} - k^{-2+m}\,(-1+m)\right)d[-1+m]\right)$$

% /. (k - 1)^m-2 → k^m-2

```
(c + k) (k^(-2+m) d[-2 + m] + k^(-1+m) d[-1 + m]) -
  (-1 + c + k + m) (k^(-2+m) d[-2 + m] + (k^(-1+m) - k^(-2+m) (-1 + m)) d[-1 + m])
```

Expand[%]

```
k^(-2+m) d[-2 + m] - k^(-2+m) m d[-2 + m] + k^(-2+m) d[-1 + m] - c k^(-2+m) d[-1 + m] -
  2 k^(-2+m) m d[-1 + m] + c k^(-2+m) m d[-1 + m] + k^(-2+m) m^2 d[-1 + m]
```

We get a polynomial of $m-2$ degree, though p is of $m-1$ degree.

Problem 5

Let a be a parameter and

$$p_n = \binom{2n}{n} a^n$$

Find all values of a such that p_k is Gosper-summable.

```
Clear[p, a, q, r, f];
s[k_] := Binomial[2 k, k] a^k
s[k + 1]/s[k] // FunctionExpand
```

$$\frac{2\,a\,(1+2\,k)}{1+k}$$

```
p[k_] := 1
q[k_] := 2 a (2 k - 1)
r[k_] := k
p[k + 1]/p[k] q[k + 1]/r[k + 1] // Simplify
```

$$\frac{2\,a\,(1+2\,k)}{1+k}$$

```
Table[PolynomialGCD[q[k], r[k + j]], {j, 0, 15}]
```

{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Note, that the order of these polynomials depends on a

```
Collect[q[k + 1] + r[k], k]
```

2 a + (1 + 4 a) k

```
Collect[q[k + 1] - r[k], k]
```

2 a + (-1 + 4 a) k

Let us assume that $a \neq 1/4$ and $a \neq -1/4$, then

```
Exponent[q[k + 1] - r[k], k] ≥ Exponent[q[k + 1] + r[k], k]
```

```
True
```

It falls into Case 1, with a polynomial solution of the degree

$$\deg(f_k) = \deg(p_k) - \deg(q_{k+1} - r_k)$$

```
Exponent[p[k], k] - Exponent[q[k + 1] - r[k], k]
```

```
-1
```

This case holds for $a \neq -1/4$.

Consider $a = 1/4$.

```
Clear[p, q, r, f];
a = 1 / 4;
s[k_] := Binomial[2 k, k] a^k
s[k + 1]/s[k] // FunctionExpand
```

$$\frac{1 + 2\,k}{2\,(1 + k)}$$

```
p[k_] := 1
q[k_] := 2 k - 1
r[k_] := 2 k
```

```
Table[PolynomialGCD[q[k], r[k + j]], {j, 0, 15}]
```

```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

```
q[k + 1] f[k] - r[k] f[k - 1] == p[k] /. f[k_] → c
```

```
-2 c k + c (-1 + 2 (1 + k)) == 1
```

```
Solve[%, c]
```

```
{{c → 1}}
```

```
z[k_] := r[k]/p[k] s[k] 1
```

The result of summation

```
z[k]
```

$2^{1-2k}\, k\, \text{Binomial}[2k, k]$