15-354: CDM	**K. Sutner**
Assignment 6	Due: **November 3, 2023, 24:00**.

1. Equational Theories (30)

Background

Suppose we have a language of first-order logic that has only function symbols (plus equality). In this case the only atomic formulae look like

$$f(x, g(x, y)) = h(y, x)$$

By an equational theory we mean a set Γ of equations of this kind (where one should think of all the variables as being universally quantified). Equational theories are hugely important in algebra. By a model of Γ we mean any first-order structure that satisfies all the axioms in Γ.

Task

A. Suppose we have only one unary function symbol f and a single axiom of the form $\gamma_n \equiv f^n(x) = x$, $n \geq 0$. Find all the models of γ_n up to isomorphism.

B. Find a way to express groups as an equational theory.

C. Wurzelbrunft thinks he has found an exceedingly clever equational theory that has only infinite models. What do you say?

D. Show how to define a product operation on the models of an equational theory so that, for all models M_1 and M_2, the product $M_1 \times M_2$ is another model (that depends on both M_1 and M_2)

Comment For part (B), you need to specify the language as well as the axioms. For part (C), recall that all our first-order structures are required to have at least one element. Lastly, in (D), the product $M_1 \times M_2$ has to depend on both M_1 and M_2, and it has to be useful.

2. Arithmetic Transducers (30)

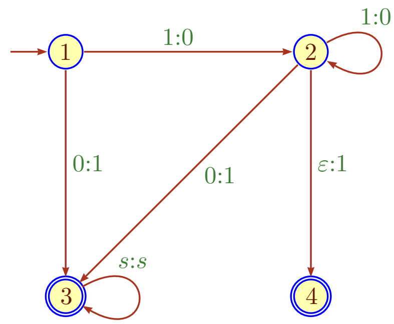
Background

In the following we will always use reverse binary representations of natural numbers, so the value of a string $x = x_0 x_1 \ldots x_{n-1} \in \mathbf{2}^\star$ is $\mathsf{val}(x) = \sum_{i<n} x_i 2^i$. We do not allow trailing zeros, except for the string '0' with value $0 : \mathbb{N}$, so numbers are represented by the regular language $N = \{0\} \cup \{0,1\}^\star 1$. Note that, restricted to N, val is a bijection.

Now suppose we have some transducer $\mathcal{T}$ defining a transduction $\tau \subseteq N \times N$. We say that $\mathcal{T}$ implements an arithmetic function $f : \mathbb{N} \to \mathbb{N}$ if

$$\tau = \{ (\mathsf{val}(x), f(\mathsf{val}(x)) \mid x \in \mathbb{N} \}$$

For example, the following transducer implements the successor function:



Task

A. Find a transducer that implements the function $n \mapsto n+2$ and prove correctness.

B. Find a transducer that implements the function $n \mapsto n+3$ and prove correctness.

C. Find a transducer that implements the function $n \mapsto 3n+2$ and prove correctness.

Comment

It may help to assume initially that there are trailing zeros.

3. Reversibility of ECA (40)

Background

Suppose $\rho : \mathbf{2}^3 \to \mathbf{2}$ is the local map of an elementary cellular automaton (i.e., a ternary Boolean function). We have seen how to construct from ρ a synchronous transducer $\mathcal{A}_{\twoheadrightarrow,x,y}$ that checks whether a finite bit sequence x evolves to y in one step under fixed boundary conditions. Naturally, there is a similar machine for cyclic boundary conditions, though things are a bit messier than in the fixed case.

Reversibility of a cellular automaton is expressed by the first-order formula

$$\mathsf{inj} \equiv \forall\, x, y, z\, (x \twoheadrightarrow z \wedge y \twoheadrightarrow z \Rightarrow x = y)$$

It is slightly easier to work with irreversibility, expressed by the negation $\mathsf{ninj} = \neg\mathsf{inj}$.

Task

A. Construct a synchronous 2-track transducer $\mathcal{B}_{\twoheadrightarrow}$ that checks whether a finite bit sequence x evolves to y in one step under cyclic boundary conditions.

B. Then build a synchronous 3-track transducer $\mathcal{A}$ that accepts the language defined by the matrix of ninj.

C. What does $\mathcal{A}$ have to do with injectivity of the global map on $\mathbf{2}^n$?

D. Explain how one can directly construct a 2-track transducer $\mathcal{A}'$ that still can be used to check ninj. This machine should be of the form $\mathcal{A}' = \mathcal{A}_0 \times \mathcal{U}$ where $\mathcal{U}$ is the un-equal transducer on 2 tracks.

Comment

For part (A), nondeterminism is critical (see the construction for the fixed boundary condition case). To avoid a silly edge case, let's assume that all words are non-empty.