CDM

State Complexity

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Where Are We?

We have seen that the class of regular languages has much stronger closure properties than one might suspect from the definition.

Even better, the proofs for these closure properties are all constructive: there are relatively straightforward algorithms to build the corresponding FSMs.

This raises the question: how effective can these FSM algorithms be? A good first step in this direction is to figure out the sizes of the machines.

But first a little algebra . . .

Homomorphisms

Recall that any map $f : \Sigma \to \Gamma^\star$ gives rise to uniquely determined monoid homomorphism $f : \Sigma^\star \to \Gamma^\star$

$$f(x) = f(x_1) f(x_2) \dots f(x_n)$$

For example,

$$f(a) = 00 \qquad f(b) = 01 \qquad f(c) = 11$$

translates from $\{a, b, c\}$ to $\mathbf{2}^\star$.

It is fairly clear that $f(L) \subseteq \mathbf{2}^\star$ is regular whenever $L \subseteq \{a, b, c\}^\star$ is. But the opposite direction is not so obvious.

Closure under Homomorphisms

Theorem

Regular languages are closed under homomorphisms and inverse homomorphisms.

Proof.

For homomorphisms, consider a NFA for $L \subseteq \Sigma^\star$ and replace transitions $p \xrightarrow{a} q$ by $p \xrightarrow{f(a)} q$.

For the opposite direction it is helpful to have a more algebraic characterization of regular languages.

Batten down the hatches.

Transduction Semigroups

Suppose we have some set X and a collection F of endofunctions on X.

Definition

(X, F) is a transduction semigroup if F is closed under composition, and a transduction monoid if F in addition contains a unit element.

If you prefer, you can think of the semigroup F as acting on X on the left in the natural way:

$$f \cdot x = f(x)$$

If we use diagrammatic composition, we get a right action.

Currying

To see the connection to finite state machines, note that we can think of the transition function of a DFA as a Σ-indexed list of functions from states to states:

$$\delta_a : Q \to Q$$
$$\delta_a(p) = \delta(p, a)$$

and we "iterate" these functions according to some input word $u = u_1 u_2 \dots u_n$:

$$\delta_u = \delta_{u_1} \circ \delta_{u_2} \circ \dots \delta_{u_{n-1}} \circ \delta_{u_n}$$

We can express acceptance as

$\mathcal{A}$ accepts a word u iff $\delta_u(q_0) \in F$.

The Algebra Perspective

This may seem like a pointless exercise, but it naturally leads to another interesting perspective: algebra. The given functions δ_a generate a transformation semigroup (monoid) T over Q, a subsemigroup of the full monoid of endofunctions $Q \to Q$.

Definition

T is called the transition semigroup (monoid) of the DFA.

One way of writing down T is

$$T = \{\, \delta_x : Q \to Q \mid x \in \Sigma^\star \,\} = \langle\, \delta_a \mid a \in \Sigma \,\rangle$$

where $\delta_x(p) = \delta(p, x)$.

Analyzing this semigroup can help quite a bit in getting a better understanding of a DFA. And, there are powerful algebraic tools available that help in dealing with the monoid.

Example 1: Even-Even

There is a natural 4-state DFA that accepts all strings over $\{a, b\}^\star$ that contain an even number of a's and an even number of b's.

p	1	2	3	4
$\delta_a(p)$	2	1	4	3
$\delta_b(p)$	3	4	1	2

The initial state is 1 and $F = \{1\}$.

But note that

$$\delta_a \circ \delta_a = I$$
$$\delta_b \circ \delta_b = I$$
$$\delta_a \circ \delta_b = \delta_b \circ \delta_a$$

so that the transformation semigroup consists of $\{I, \delta_a, \delta_b, \delta_a \circ \delta_b\}$. Note that this is actually a monoid and even a group (Kleinsche Vierergruppe).

Moreover, from the equations it is easy to see that for any word x

$$\delta_x = I \iff \#_a x \text{ even}, \#_b x \text{ even}$$

Similarly we have

$$\delta_x = \delta_a \iff \#_a x \text{ odd}, \#_b x \text{ even}$$

and so on.

At the very least this very elegant and concise.

Example 2: $L_{a,-2}$

The usual de Bruijn automaton yields

p	1	2	3	4
$\delta_a(p)$	1	3	1	3
$\delta_b(p)$	2	4	2	4

This generates the semigroup (no monoid here)

$$(1,3,1,3),(2,4,2,4),(1,1,1,1),(2,2,2,2),(3,3,3,3),(4,4,4,4)$$

Question: What do the constant functions mean?

Semigroups versus Monoids

The distinction between semigroups and monoids here is a bit of a technical nuisance, but there is no easy way to get rid of it.

At any rate, note that we can turn any semigroup $\mathcal{S}$ into a monoid $\mathcal{S}^1$ by simply adding a new element 1 and defining

$$x \cdot 1 = 1 \cdot x = x$$

for all x in $\mathcal{S}$.

Clearly, $\mathcal{S}$ and $\mathcal{S}^1$ are essentially the same.

Also note that a transition semigroup may be a monoid without containing the identity function.

Regularity and Algebra

The reason monoids are important here is because they provide a characterization of regular languages that is free of any combinatorial aspect. Always remember: algebra is the great simplifier.

Theorem

A language $L \subseteq \Sigma^\star$ is regular iff there is a finite monoid M, $M_0 \subseteq M$ and a monoid homomorphism $f : \Sigma^\star \to M$ such that $L = f^{-1}(M_0)$.

Proof.

If L is regular, let M be the transformation monoid of a DFA that recognizes L, and define $f(x) = \delta_x$ and $M_0 = \{ g \in M \mid g(q_0) \in F \}$.

More Proof

The opposite direction is more interesting: we construct a DFA

$$\mathcal{A} = \langle\, M, \Sigma, \delta; 1_M, M_0 \,\rangle$$

where $\delta(p, a) = p \cdot f(a)$. Then $\delta(p, x) = p \cdot f(x)$ and $\delta(q_0, x) = f(x)$. $\square$

Message: anything goes as a state set, as long as the set is finite. For the implementer, the state set is always $[n]$, but that's not a good way to think about it.

Algebraic automata theory is a fascinating subject with lots of elegant results, but it requires work and there is no essential algorithmic payoff. So, we won't go there.

Size of an Automaton

We already defined the state complexity of a FSM to be the number of states of the machine.

This is the standard measure of the size of a FSM and most results are phrased in terms of state complexity.

But note that this is a bit of an oversimplification: we really should be dealing with the transition complexity, the number of transitions, simply because this number corresponds more faithfully to the size of a FSM data structure.

Algorithmic Issues

So we have three increasingly complicated types of machines: DFAs, NFAs and NFAEs, that all accept exactly the regular languages. There are two conversion algorithms:

- Elimination of ε-moves: conversion from NFAE to NFA.
- Elimination of nondeterminism: conversion from NFA to DFA.

The first one is comes down to computing transitive closure of the ε-transitions and can be handled efficiently using standard graph algorithms.

But nondeterminism is more difficult to get rid of: there may be an exponential blow-up in the state complexity of the deterministic machine.

Epsilon Elimination

Given an NFAE $\mathcal{A}$ of state complexity n, the first step in ε-elimination is to compute the ε-closures of all states; this takes at most $O(n^3)$ steps.

Introducing new transitions preserves state complexity, but can increase the transition complexity by a quadratic factor.

One interesting implementation idea for pattern matching is not to pre-process all of $\mathcal{A}$: instead one computes the closures on the fly and only when actually needed. This may be faster if the machine is large and only used a few times.

Exponential Blow-Up

Alas, the powerset construction is potentially exponential in the size of $\mathcal{A}$, even when only the accessible part pow$(\mathcal{A})$ is constructed. The only general bound for the state complexity of pow$(\mathcal{A})$ is 2^n.

In practice, it happens quite often that the accessible part is small, but there are cases when the state complexity of the deterministic machine is close to 2^n. Even worse, it can happen that this large power automaton is already minimal, so there is no way to get rid of these exponentially many states.

Incidentally, determinization is quite fast as long as the resulting automaton is not too large.

Blow-Up Example 1

Recall the kth symbol languages

$$L(a,k) = \{\, x \mid x_k = a \,\}$$

Proposition

$L(a,-k)$ can be recognized by an NFA on $k+1$ states, but the state complexity of this language is 2^k.

Proof.

We have already seen the de Bruijn DFA for the language, a machine of size 2^k (at least for a binary alphabet).

It remains to show that this machine already has the smallest possible number of states.

Minimality

Suppose $\mathcal{A}$ is a DFA for $L_{0,-k}$ on less than 2^k states.

Consider all 2^k inputs $x \in \mathbf{2}^k$ and let

$$p_x = \delta(q_0, x)$$

Then $p_x = p_y$ for some $x \neq y$.

But then there is a word u such that $xu \in L_{0,-k}$ and $yu \in L_{0,-k}$.

Contradiction.

$\square$

Weird Asymmetry

We know already that

$$L \text{ is regular} \iff L^{\mathrm{op}} \text{ is regular}$$

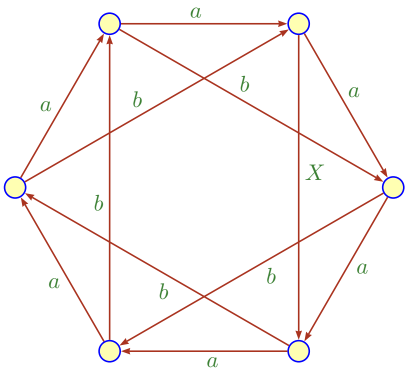
But there may be a price to pay: the state complexity of L^{op} may be exponential in the state complexity of L.

Blow-Up Example 2

Here is a 6-state NFA based on a circulant graph. Assume $I = F = Q$.

If $X = b$ than the power automaton has size 1.

However, for $X = a$ the power automaton has maximal size 2^6.

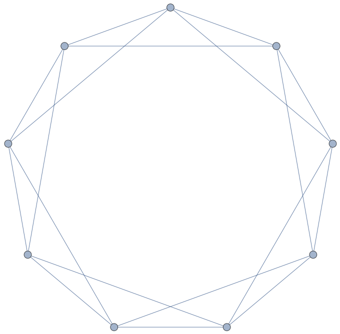


Circulants

The example generalizes to a whole group of circulant machines on n states with diagram $C(n;1,2)$.

These machines are based on circulant graphs:

Vertices	$\{0,1,\dots,n-1\}$
Edges	$(v, v+1 \bmod n)$ and $(v, v+2 \bmod n)$



Tip of an Iceberg

Start with a labeling where the edges with stride 1 are labeled a and the edges with stride 2 are labeled b.

Then change exactly one of these edge labels: the resulting nondeterministic machines have power automata of size 2^n and the power automata are already minimal.

Exercise

Prove that full blow-up occurs for all these NFA.

Exercise

How about other circulants $C(n; 1, k)$*?*

The Argument

In general, we have some nondeterministic automaton $\mathcal{A} = \langle Q, \Sigma, \tau; Q, Q \rangle$ and we want to show that pow$(\mathcal{A})$ has size 2^n (or some such).

This boils down to the following. Let $\delta_a : \mathfrak{P}(Q) \to \mathfrak{P}(Q)$ be the function induced by the relation τ, as in the determinization algorithm:

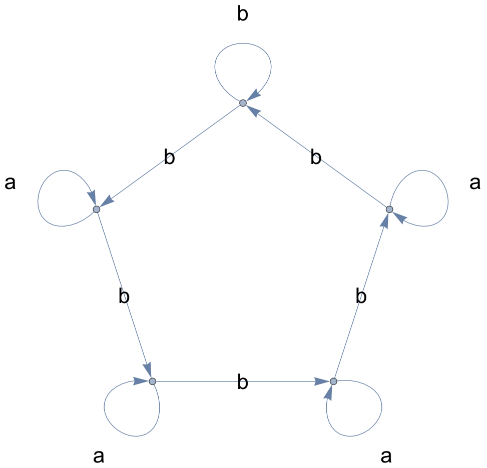
$$\delta_a(P) = \{ q \in Q \mid \exists\, p \in P \;\tau(p, a, q) \}$$

By composition we get the transition semigroup, functions δ_x for all $x \in \Sigma^\star$. For each $P \subseteq Q$ we need to find a witness $x \in \Sigma^\star$ such that

$$\delta_x(Q) = P$$

Easy Case

Consider $C(n; 0, 1)$, label all loops a and all stride 1 edges b. Then switch the label of the loop at 0.



Proof?

δ_a kill 0

δ_b sticky rotate

Note that $Q = \{0, 1, \ldots, n-1\}$ is reachable from any $P \neq \emptyset$.

If we can concoct the operation "rotate" we are done.

Case 1: $0 \notin P$ δ_b works

Case 2: $0 \in P$

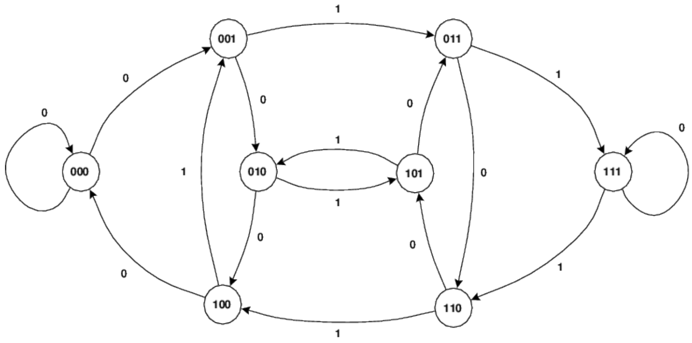
If $n-1 \in P$: δ_b works

If $n-1 \notin P$: ????

Another Blow-Up

Start with a binary de Bruijn semiautomaton where both δ_0 and δ_1 are permutations (so that the transition semigroup is actually a subgroup of the symmetric group on Q). Now flip the label of the loop at $\mathbf{0}$.

For $I = F = Q$, full blow-up occurs.



A Little Challenge

The loop case I can prove. But here is an open problem:

One can show that the number of permutation labelings in the binary de Bruijn graph of rank k is $2^{2^{k-1}}$. Flipping the label of an arbitrary edge will produce full blow-up in exactly half of the cases.

$$\text{full blow-up:} \qquad 2^k \, 2^{2^{k-1}}$$

Verified experimentally up to $k = 5$ (on Blacklight at PSC, rest in peace). There are $8,388,608$ machines to check, ignoring symmetries.

Predicting Blow-Up

Many finite state machine algorithms naturally produce nondeterministic machines. Exponential blow-up makes it somewhat difficult to decide whether it is advantageous to compute the corresponding power automaton: the actual matching process is faster but the machine may be too large.

Alas, we cannot predict how big the deterministic machine will be: the following problem is PSPACE-complete.

Problem:	**Power Automaton Size**
Instance:	A nondeterministic machine $\mathcal{A}$, a bound B.
Question:	Is the size of the power automaton of $\mathcal{A}$ bounded by B?

Lower Bounds

In general one would like to have estimates for the size of a machine constructed by a certain algorithm, as a function of the size of the input machines.

Upper bounds are usually easy to get, they are quite obvious from the constructions.

But lower bounds are tricky; one needs to construct particular inputs that make the algorithm perform poorly.

Note that this is just worst-case analysis, very little is currently known about average-case behavior.

State Complexity of Operations

	DFA	NFA
intersection	mn	mn
union	mn	$m+n$
concatenation	$(m-1)2^n-1$	$m+n$
Kleene star	$3 \cdot 2^{n-2}$	$n+1$
reversal	2^n	n
complement	n	2^n

Worst potential blow-up starting from machine(s) of size m, n and applying the corresponding operation.

Note that we are only dealing the state complexity, not transition complexity.

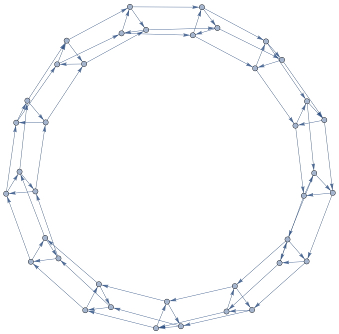
Example: Intersection

Let

$$K_{a,m} = \{\, x \in \mathbf{2}^\star \mid \#_a x = 0 \pmod{m} \,\}$$

be the “mod-counter” language. Clearly $K_{a,m}$ has state complexity m.

The intersection of $K_{0,m}$ and $K_{1,n}$ has state complexity mn.



Decision Problems

- The seminal 1959 paper by Rabin and Scott also introduced the study of the computational complexity of various decision problems associated with finite state machines.
- We have already seen some of these: Emptiness, Finiteness, Universality, Equality and Inclusion. For DFAs they are all easily solvable (linear or quadratic time).
- For NFAs the situation is more complicated (NFAEs are not relevant here since they can be converted to NFAs in cubic time).

Decision Problems for Regular Languages

Problem:	**Emptiness Problem**
Instance:	A regular language L.
Question:	Is L empty?

Problem:	**Finiteness Problem**
Instance:	A regular language L.
Question:	Is L finite?

Problem:	**Universality Problem**
Instance:	A regular language L.
Question:	Is $L = \Sigma^\star$?

Ambiguity

Note that these problems are all slightly ambiguous as stated: exactly how is the the input (a regular language) given?

As far as decidability is concerned there is no difference between DFAs and NFAs: we can simply convert the NFA.

But the determinization may be exponential, so efficiency becomes a problem.

- Emptiness and Finiteness are easily polynomial time for DFAs and NFAs.
- Universality is polynomial time for DFAs but PSPACE-complete for NFAs.

More Problems

Problem:	**Equality Problem**
Instance:	Two regular languages L_1 and L_2.
Question:	Is L_1 equal to L_2?

Problem:	**Inclusion Problem**
Instance:	Two regular languages L_1 and L_2.
Question:	Is L_1 a subset of L_2?

- Equality and Inclusion are polynomial time for DFAs.
- Both problems are PSPACE-complete for NFAs.

Weird Automata

In a DFA there is exactly one trace for each input.

In an NFA there may be exponentially many, and acceptance is determined by an existential quantification: is there a run . . . . So here is a tempting question:

> Is there a useful notion of acceptance based on
> "for all runs such that such and such"?

One problem is whether these "universal" automata are more powerful than ordinary FSMs. As we will see, we only get regular languages.

Of course, this raises the question of how the state complexities compare.

Forall Automata

How would one formally define a type of FSM where acceptance means all runs have a certain property?

The underlying transition system will be unaffected, it is still a labeled digraph.

The acceptance condition now reads:

$\mathcal{A}$ accepts x if all runs of $\mathcal{A}$ on x starting at I end in F.

Let's call these machines ∀FA .

Read: universal FA. Actually, don't: this collides with our previous use where universal means "accepting all inputs." Just look at the beautiful symbol and don't say anything. Wittgenstein would approve.

By the same token, a NFA would be a ∃FA.

Logical And

As an example consider again the mod counter languages

$$K_{a,m} = \{\, x \in \mathbf{2}^\star \mid \#_a\, x = 0 \pmod{m} \,\}$$

with state complexity m. For the union $K_{0,m} \cup K_{1,n}$ we have a natural NFA of size $m + n$. However, for the intersection $K_{0,m} \cap K_{1,n}$ we only have a product machine that has size mn.

More importantly, note that nondeterminism does not seem to help with intersection: there is no obvious way to construct a smaller NFA for $K_{0,m} \cap K_{1,n}$.

But we can build a ∀FA of size just $m + n$: take the disjoint union and declare the acceptance condition to be universal.

A Hidden Product

What is really going on here?

Let's assume that Q_1 and Q_2 are disjoint. Starting at $\{q_{01}, q_{02}\}$ we update both components. So after a while we are in state

$$\{p, q\}$$

where $p \in Q_1$ and $q \in Q_2$. In the end we accept iff $p \in F_1$ and $q \in F_2$.

This is really no different from a product construction, we just don't spell out all the product states explicitly. Choosing clever representations is sometimes critically important. Trust me.

That's Not Insane

Note that acceptance testing for a ∀FA is no harder than for an NFA: we just have to keep track of the set of states $\delta(I, x) \subseteq Q$ reachable under some input and change the notion of acceptance: this time we want $\delta(I, x) \subseteq F$.

For example if some word x crashes all possible computations so that $\delta(I, x) = \emptyset$ then x is accepted.

Likewise we can modify the Rabin-Scott construction that builds an equivalent DFA: as before calculate the (reachable part of the full) powerset and adjust the notion of final state:

$$F' = \{\, P \subseteq Q \mid P \subseteq F \,\}$$

Wisdom

Good mathematicians see analogies between theorems or theories; the very best ones see analogies between analogies.

S. Banach

An Analogy (pace Banach)

We can think of the transitions in a NFA as being disjunctions:

$$\delta(p, a) = q_1 \vee q_2$$

We can pick q_1 or q_2 to continue. Likewise, in a ∀FA we are dealing with conjunctions:

$$\delta(p, a) = q_1 \wedge q_2$$

meaning: We must continue at q_1 and at q_2. So how about

$$\delta(p, a) = (q_1 \vee q_2) \wedge (q_3 \vee q_4)$$

Or perhaps

$$\delta(p, a) = (q_1 \vee \neg q_2) \wedge q_3$$

Does this make any sense?

Threads

Think of threads: both $\wedge$ and $\vee$ correspond to launching multiple threads. The difference is only in how we interpret the results returned from each of the threads.

For $\neg$ there is only one thread, and we flip the answer bit.

In other words, a “Boolean” automaton produces a computation tree rather than just a single branch. This is actually not much more complicated than an NFA.

For historical reasons, these devices are called alternating automata.

Alternating Automata

In an alternating automaton (AFA) we admit transitions of the form

$$\delta(p, a) = \varphi(q_1, q_2, \ldots, q_n)$$

where φ is an arbitrary Boolean formula over Q, even one containing negations.

How would such a machine compute? Initially we are in “state”

$$q_{01} \vee q_{02} \vee \ldots \vee q_{0k}$$

the disjunction of all the initial states.

One Step and Acceptance

Suppose we are in state Φ, some Boolean formula over Q. Then under input a the next state is

$$\Phi[p_1 \mapsto \delta(p_1, a), \ldots, p_n \mapsto \delta(p_n, a)]$$

Thus, all variables $q \in Q$ are replaced by $\delta(q, a)$, yielding a new Boolean formula. In the end we accept if

$$\Phi[F \mapsto 1, \overline{F} \mapsto 0] = 1$$

Exercise

Verify that for NFA and $\forall$FA this definition behaves as expected.

Why Alternating?

The name “alternating automaton” may sound a bit strange.

The original paper by Chandra, Kozen and Stockmeyer that introduced these machines in 1981 showed that one can eliminate negation without reducing the class of languages.

One can then think of alternating between existential states (at least one spawned process must succeed) and universal states (all spawned processes must succeed).

Also note that alternation makes sense for other machines. For example, one can introduce alternating Turing machines (quite important in complexity theory).

Theorem

Alternating automata accept only regular languages.

Proof.

We can build a DFA over the state set $\mathsf{Bool}(Q)$ (the collection of all Boolean formulae with variables in Q) where one representative is chosen in each class of equivalent formulae (so there are at most 2^{2^n} states). For example, we could choose conjunctive normal form.

The transitions are described as above, except that after the substitution we must bring the formula back into the chosen normal form.

The final states are the ones that evaluate to true as above.

□

Why Would Anyone Care?

Because an AFA can be much, much smaller that the minimal DFA. In fact, the 2^{2^n} bound is tight: there are AFAs on n states where the minimal equivalent DFA is doubly exponential in n.

So we have a very concise representation for a regular language but one that still behaves reasonably well under the usual algorithms. Avoiding the standard DFA representation is often critical for feasibility: in reality we cannot actually construct the full DFA. Laziness is a good idea in this context.

BTW, this is even true in pattern matching: DFAs should be avoided unless they are absolutely necessary (because the pattern contains a negation).