CDM
Register Machines

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1. More Recursion

- Register Machines
- Coding
- Universality

Recall: Computability

Recall that we are looking for a formal definition of computability. So far we have encountered primitive recursive functions. We have seen that primitive recursive functions provide a huge supply of intuitively computable functions. So could this be the final answer? Sadly, NO ...

- There are clearly computable functions (based on a more general type of recursion) that fail to be primitive recursive.
- Computability forces functions to be partial in general, we need to adjust our framework correspondingly.

In primitive recursion one often encounters terms like $x + 1$.

This is fine, but one can remove a bit a visual clutter by writing

$$x^+$$

And, of course, x^{++} stands for $x + 2$, and so on.

The key idea behind primitive recursive functions is to formalize a certain type of recursion (or, if you prefer, inductive definition). As we have seen, even some extension of this basic idea like course-of-value recursion remains within this framework.

Here is another example: simultaneous recursion. To simplify matters, let us only consider two simultaneously defined functions as in

$$f_i(0, \mathbf{y}) = g_i(\mathbf{y})$$

$$f_i(x^+, \mathbf{y}) = h_i(x, f_1(x, \mathbf{y}), f_2(x, \mathbf{y}), \mathbf{y})$$

where $i = 1, 2$.

$$f_1(0) = 1$$
$$f_1(x^+) = f_2(x)$$

$$f_2(0) = 0$$
$$f_2(x^+) = f_1(x)$$

Then f_1 is the characteristic function of the even numbers. We'll see better examples later when we talk about context free grammars.

Lemma

Simultaneous recursion is admissible for primitive recursive functions.

Ackermann's Function (1928)

And yet, there is more to recursion that this. Define the Ackermann function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by the following double recursion (this version is actually due to Rózsa Péter and simpler than Ackermann's original function):

$$A(0, y) = y^+$$
$$A(x^+, 0) = A(x, 1)$$
$$A(x^+, y^+) = A(x, A(x^+, y))$$

On the surface, this looks more complicated than primitive recursion. Of course, it is not at all clear that Ackermann could not somehow be turned into a p.r. function.

Here is a bit of C code that implements the Ackermann function (assuming that we have infinite precision integers).

```
int acker(int x, int y)
{
  return( x ? (acker(x-1, y ? acker(x, y-1) : 1)) : y+1 );
}
```

All the work of organizing the nested recursion is handled by the compiler and the execution stack. So this provides overwhelming evidence that the Ackermann function is intuitively computable.

We could memoize the values that are computed during a call to $A(a, b)$.

It is not hard to see that whenever $A(x, y)$ is in the hash table, so is $A(x, z)$ for all $z < y$ (except for $x = 0$). Think of the table as having x rows.

But note that the computation of $A(a, b)$ will require entries $A(a - 1, z)$ for $z > b$. So each row in the table gets longer and longer as x gets smaller.

For example, the computation of $A(3, 3) = 61$ requires $A(2, 29)$ and $A(1, 59)$.

Note that the hash table provides a proof that $A(3, 3) = 61$.

It is useful to think of Ackermann's function as a family of unary functions $(A_x)_{x\geq 0}$ where $A_x(y) = A(x, y)$ ("level x of the Ackermann hierarchy").

The definition then looks like so:

$$A_0(y) = y^+$$

$$A_{x^+}(0) = A_x(1)$$

$$A_{x^+}(y^+) = A_x(A_{x^+}(y))$$

From this it follows easily by induction that

Lemma

Each of the functions A_x is primitive recursive (and hence total).

$$A(0, y) = y^+$$

$$A(1, y) = y^{++}$$

$$A(2, y) = 2y + 3$$

$$A(3, y) = 2^{y+3} - 3$$

$$A(4, y) = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} - 3$$

The first 4 levels of the Ackermann hierarchy are easy to understand, though A_4 starts causing problems: the stack of 2's in the exponentiation has height $y + 3$.

This usually called super-exponentiation or tetration and often written ${}^y x$ or $x{\uparrow\uparrow}y$.

Alas, if we continue just a few more levels, darkness befalls.

$$A(5,y) \approx \text{super-super exponentiation}$$

$$A(6,y) \approx \text{an unspeakable horror}$$

$$A(7,y) \approx \text{speechless}$$

For level 5, one can get some vague understanding of iterated super-exponentiation, but things start to get murky.

At level 6, we iterate over the already nebulous level 5 function, and things really start to fall apart.

At level 7, Wittgenstein comes to mind: "Wovon man nicht sprechen kann, darüber muss man schweigen."

One might think that the only purpose of the Ackermann function is to refute the claim that computable is the same as p.r. Surprisingly, the function pops up in the analysis of the Union/Find algorithm (with ranking and path compression).

The running time of Union/Find differs from linear only by a minuscule amount, which is something like the inverse of the Ackermann function.

But in general anything beyond level 3.5 of the Ackermann hierarchy is irrelevant for practical computation.

Exercise

Read an algorithms text that analyzes the run time of the Union/Find method.

Theorem

The Ackermann function dominates every primitive recursive function f in the sense that there is a k such that

$$f(\mathbf{x}) < A(k, \max \mathbf{x}).$$

Hence A is not primitive recursive.

Sketch of proof.

One can argue by induction on the buildup of f.

The atomic functions are easy to deal with.

The interesting part is to show that the property is preserved during an application of composition and of primitive recursion. Alas, the details are rather tedious.

□

Informally, the Ackermann function cannot be primitive recursive because it grows far too fast. On the other hand, it does not really have a particular purpose other than that.

We will give another example of mind-numbing growth based on actual counting problem. To this end, it is easier to use a slight variant of the Ackermann function.

$$B_1(x) = 2x$$

$$B_{k^+}(x) = B_k^x(1)$$

So B_1 is doubling, B_2 exponentiation, B_3 super-exponentiation and so on.

In general, B_k is closely related to A_{k+1}.

Subsequence Order

Recall the subsequence ordering on words where $u = u_1 \dots u_n$ precedes $v = v_1 v_2 \dots v_m$ if there exists a strictly increasing sequence $1 \leq i_1 < i_2 < \dots i_n \leq m$ of positions such that $u = v_{i_1} v_{i_2} \dots v_{i_n}$.

In symbols: $u \sqsubseteq v$.

In other words, we can erase some letters in v to get u.

Subsequence order is not total unless the alphabet has size 1.

Note that subsequence order is independent of any underlying order of the alphabet (unlike, say, lexicographic or length-lex order).

Warmup: Antichains

An antichain in a partial order is a sequence x_0, x_1, ..., x_n, ... of elements such that $i < j$ implies that x_i and x_j are incomparable.

Example

Consider the powerset of $[n] = \{1, 2, \ldots, n\}$ with the standard subset ordering. How does one construct a long antichain?

For example, $x_0 = \{1\}$ is a bad idea, and $x_0 = [n]$ is even worse.

What is the right way to get a long antichain?

Theorem (Higman's 1952)

Every antichain in the subsequence order is finite.

Proof. Here is the Nash-Williams proof (1963): assume there is an antichain.

For each n, let x_n be the length-lex minimal word such that $x_0, x_1, \ldots, x_n$ starts an antichain, producing a sequence $x = (x_i)$.

For any sequence of words $y = (y_i)$ and a letter a define $a^{-1}y$ to be the sequence consisting of all words in y starting with letter a, and with a removed.

Since the alphabet is finite, there exists a letter a such that $x' = a^{-1}x$ is an antichain. But then $x'_0 < x_0$, contradiction.

□

For a finite or infinite word x write $x[i]$ for the block $x_i, x_{i+1}, \ldots, x_{2i}$. We will always assume that $i \leq |x|/2$ when x is finite.

Bizarre Definition: A word is self-avoiding if for $i < j$ the block $x[i]$ is not a subsequence of $x[j]$.

The following is an easy consequence of Higman's theorem.

Theorem

Every self-avoiding word is finite.

Write Σ_k for an alphabet of size k.

By the last theorem and König's lemma, the set S_k of all finite self-avoiding words over Σ_k must itself be finite.

But then we can define the following function:

$$\alpha(k) = \max\big(|x| \mid x \in S_k \big)$$

So $\alpha(k)$ is the length of the longest self-avoiding word over Σ_k.

Note that α is intuitively computable: we can build the tree of all self-avoiding words by brute-force.

Trivially, $\alpha(1) = 3$.

A little work shows that $\alpha(2) = 11$, as witnessed by $abbbaaaaaaa$.

But

$$\alpha(3) > B_{7198}(158386),$$

an incomprehensibly large number.

Smelling salts, anyone?

It is truly surprising that a function with as simple a definition as α should exhibit this kind of growth.

Functions like A, B or α are all intuitively computable, but fail to be primitive recursive. So how much do we have to add to primitive recursion to capture these functions?

Proposition (Unbounded Search)

There is a primitive recursive relation R such that

$$A(a,b) = (\min\big(\, z \mid R(a,b,z)\,\big))_0$$

Think of z as a pair $\langle\, c, t\,\rangle$ where t encodes a data strucure that represents the whole computation of the Ackermann function on input a and b, something like a huge memoization table. c is the final result of this computation.

Here is a much better description.

A computation of $A(2,1)$ might start like

$$A(2,1) = A(1, A(2,0)) = A(1, A(1,1)) = A(1, A(0, A(1,0))) = \ldots$$

Note that the A's and parens are just syntactic sugar, a better description would be

$$2,1 \rightsquigarrow 1,2,0 \rightsquigarrow 1,1,1 \rightsquigarrow 1,0,1,0 \rightsquigarrow 1,0,0,1 \rightsquigarrow 1,0,2 \rightsquigarrow 1,3 \rightsquigarrow 0,1,2$$
$$\rightsquigarrow 0,0,1,1 \rightsquigarrow 0,0,0,1,0 \rightsquigarrow 0,0,0,0,1 \rightsquigarrow 0,0,0,2 \rightsquigarrow 0,0,3 \rightsquigarrow 0,4 \rightsquigarrow 5$$

We can recover the original expression by applying A like a right-associative binary operator to these argument lists.

Clearly, we can model these steps by a function Δ defined on sequences of naturals–or, via coding, just naturals.

$$\Delta(\ldots, 0, y) = (\ldots, y^+)$$

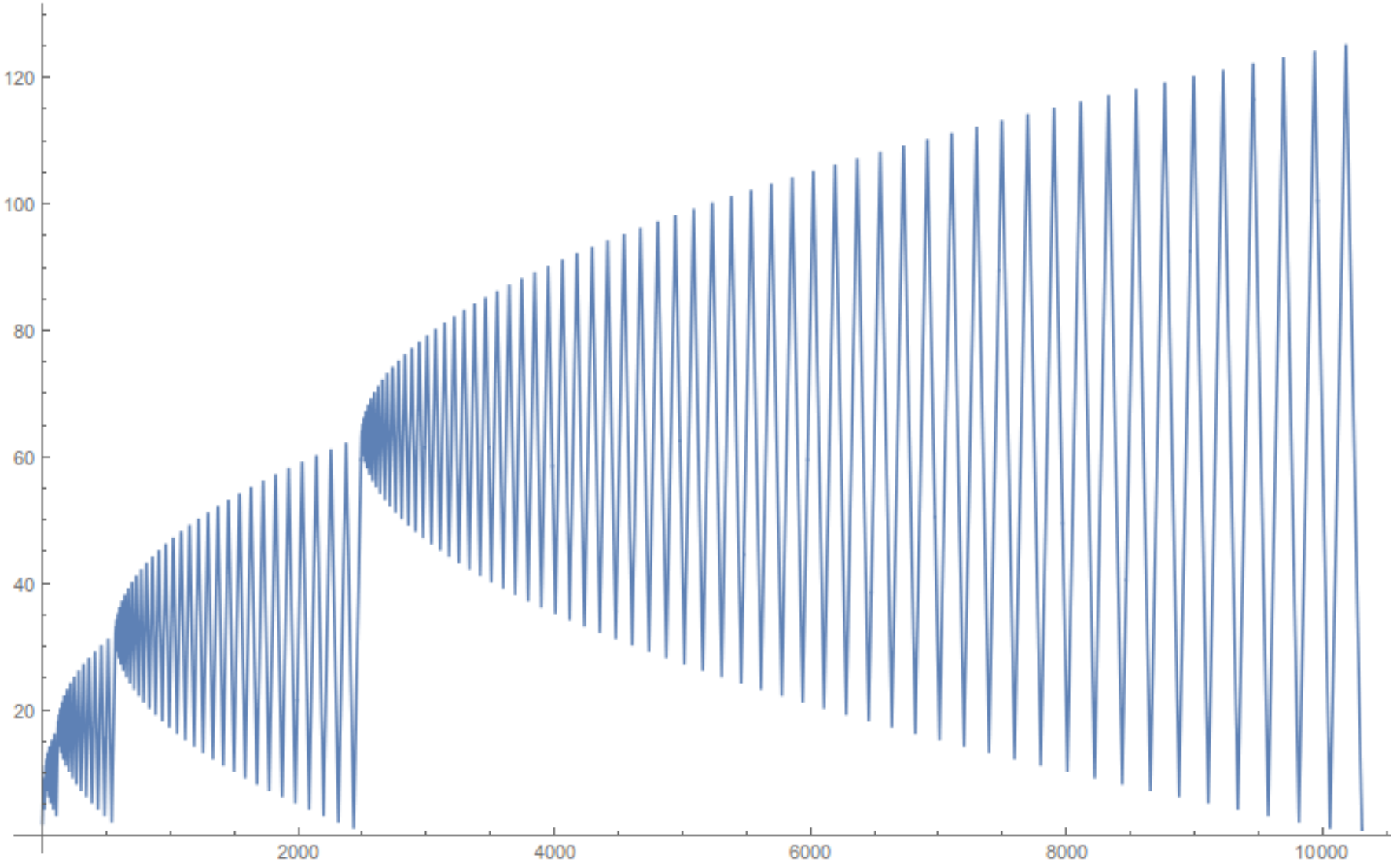
$$\Delta(\ldots, x^+, 0) = (\ldots, x, 1)$$

$$\Delta(\ldots, x^+, y^+) = (\ldots, x, x^+, y)$$

Since A is total, we know that for any a and b there is some time t such that

$$\Delta^t(a, b) = (c)$$

Clearly this condition is primitive recursive in (a, b, c, t).


120
100
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8000
10000

The diagonal function $A(x, x)$ fails to be primitive recursive. But

Claim

The predicate "$A(x, x) = z$" is primitive recursive.

Proof.

We have $A(x, y) > y$.

But then $\Delta(L) \geq L_i + i - 1$. □

Here is another obstacle to capturing computability: recall the evaluation operator for our PR terms:

$$\text{eval}(\tau, \mathbf{x}) = \text{ value of } \tau^\star \text{ on input } \mathbf{x}$$

It is clear that eval is intuitively computable (take a compilers course). In fact, it is not hard to implement in eval in any modern programming language.

Question: Could eval be primitive recursive?

A useless answer would be to say **no**, the types don't match.

The first argument of eval is a term τ in our PR language, so our first step will be to replace τ by an index $\widehat{\tau} \in \mathbb{N}$.

The index $\widehat{\tau}$ will be constructed in a way that makes sure that all the operations we need on indices are clearly primitive recursive.

The argument vector $\mathbf{x} \in \mathbb{N}^n$ will also be replaced by its sequence number $\langle x_1, \ldots, x_n \rangle$. Hence we will be able to interpret eval as a function of type

$$\mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

and this function might potentially be primitive recursive.

$$
\begin{array}{rr}
0 & \langle\, 0, 0 \,\rangle \\
P_i^n & \langle\, 1, n, i \,\rangle \\
S & \langle\, 2, 1 \,\rangle \\
\mathsf{Prec}[h, g] & \left\langle\, 3, n, \widehat{h}, \widehat{g} \,\right\rangle \\
\mathsf{Comp}[h, g_1, \ldots, g_n] & \left\langle\, 4, m, \widehat{h}, \widehat{g_1}, \ldots, \widehat{g_n} \,\right\rangle
\end{array}
$$

Thus for any index e, the first component $(e)_0$ indicates the type of function, and $(e)_1$ indicates the arity.

There is nothing sacred about this way of coding PR terms, there are many other, equally natural ways.

Now suppose eval is p.r., and define the following function

$$f(x) := \mathsf{eval}(x, x) + 1$$

Then f is also p.r. and must have an index e. But then

$$f(e) = \mathsf{eval}(e, e) + 1 = f(e) + 1$$

and we have a contradiction.

So eval is another example of an intuitively computable function that fails to be primitive recursive.

This example may be less sexy than the Ackermann function, but it appears in similar form in other contexts.

More precisely, given a good formal definition of computability (one that matches our intuitive ideas), we would still expect to have indices for the functions in this class.

We also would expect to have something like an operation

$$\text{eval} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

that works like an interpreter for functions in this class.

Most importantly, we want eval to be itself computable.

In light of the last result, this may sound like mission impossible, but we will see that things work out fine as long as we are willing to deal with partial functions.

- More Recursion

2. Register Machines

- Coding

- Universality

We could try to deal with the Ackermann function and evaluation by strengthening the machinery used to define primitive recursive functions. In particular a stronger recursion scheme seems promising.

Let's delay this approach for the moment, and instead turn a machine model, another critical method to define computability and complexity classes. There are many plausible approaches, we'll start with a model that is slightly reminiscent of assembly language programming, only that our language is much, much simpler than real assembly languages.

Again, there are other, substantially different ways to define computability–this is a good sign, it means we have a natural and robust concept.

- J. Herbrand, K. Gödel: systems of equations.
- A. Church: λ-calculus.
- A. Turing: Turing machines.
- S. C. Kleene: μ-recursive functions.
- E. Post: production systems.
- A. A. Markov: Markov algorithms (string rewriting).
- Z. Manna: while programs.

Gödel was dissatisfied with Church's early attempts to capture computability in terms of λ-definability. After Church switched to recursive functions, there was still not much positive reaction from Gödel; he was probably quite aware of the persistent weakness in the attack on the Entscheidungsproblem.

But to Turing's work Gödel responded most enthusiastically.

> *This concept, . . . is equivalent to the concept of a "computable function of integers" . . . The most satisfactory way, in my opinion, is that of reducing the concept of finite procedure to that of a machine with a finite number of parts, as has been done by the British mathematician Turing.*

Gödel always gave full credit to Turing, never to Church or himself.

Why Not Turing Machines?

Turing machines naturally compute partial functions on words:

$$f : \Sigma^\star \hookrightarrow \Sigma^\star$$

That is very convenient for complexity theory, but slightly less so for arithmetic functions

$$f : \mathbb{N}^k \hookrightarrow \mathbb{N}$$

For this to work we need to code natural numbers as strings.

We can avoid coding entirely by using a machine model that manipulates numbers directly.

Definition

A register machine (RM) consists of a finite number of registers and a control unit.

We write R_0, R_1, ... for the registers and $[R_i]$ for the content of the ith register: a single natural number.

Note: there is no bound on the size of the numbers stored in our registers, any number of bits is fine. This is where we break physics.

The control unit is capable of executing certain instructions that manipulate the register contents.

Our instruction set is very, very primitive:

- `inc r k`
  increment register R_r, goto k.
- `dec r k l`
  if $[R_r] > 0$ decrement register R_r and goto k, otherwise goto l.
- `halt`
  well ...

The gotos refer to line numbers in the program; note that there is no indirect addressing. These machines are sometimes called counter machines.

Definition

A register machine program (RMP) is a sequence of RM instructions $P = I_0, I_1, \ldots, I_{n-1}$.

For example, the following program performs addition:

```
// addition   R0 R1 --> R2
  0:   dec 0  1  2
  1:   inc 2  0
  2:   dec 1  3  4
  3:   inc 2  2
  4:   halt
```

Since we have no intentions of actually building a physical version of a register machine, this distinction between register machines and register machines programs is slightly silly.

Still, it's good mental hygiene: we can conceptually separate the physical hardware that supports some kind of computation from the programs that are executed on this hardware. For real digital computers this makes perfect sense. A similar problem arises in the distinction between the syntax and semantics of a programming language.

And, it leads to the juicy question: what is the relationship between physics and computation? We'll have more to say about this in a while.

Definition

A function is RM-computable if there is some RMP that implements the function.

This is a bit wishy-washy: we really need to fix

- a register machine program P,
- input registers I, and
- output registers O.

Then (P, I, O) determines a partial function $f : \mathbb{N}^k \hookrightarrow \mathbb{N}^\ell$ where $k = |I|$ and $\ell = |O|$.

Reasonable I/O Conventions

- Given input arguments $\mathbf{a} = (a_0, \ldots, a_{k-1}) \in \mathbb{N}^k$, set the input registers: $[R_i] = a_i$.
- All other registers in P are initialized to 0.
- Then run the program.
- If it terminates, read off the values of R_j, $j \in O$, producing the result $\mathbf{b} = (b_1, \ldots, b_\ell) = f(\mathbf{a})$.
- If P does not terminate, $f(\mathbf{a})$ is undefined.

Note the complication: we have to deal with the possibility that program P does not produce any value, in which case $f(\mathbf{a})$ is undefined.

This causes a number of difficulties. For example, what should equality mean in

$$f(\mathbf{a}) = g(\mathbf{a})$$

Alas, there is no way around this. Computability and partial functions are inseparable.

To describe a computation of P we need to explain what a snapshot of a computation is, and how get from one snapshot to the next. Clearly, for RMPs we need two pieces of information:

- the current instruction, and
- the contents of all registers.

Definition

A configuration of P is a pair $C = (p, \mathbf{x}) \in \mathbb{N} \times \mathbb{N}^n$.

Steps in a Computation

Configuration $(p, \mathbf{x})$ evolves to configuration $(q, \mathbf{y})$ in one step under P if

- $I_p = \texttt{inc r k}$:
  $q = k$ and $\mathbf{y} = \mathbf{x}[x_r \mapsto x_r + 1]$.

- $I_p = \texttt{dec r k l}$:
  $x_r > 0$, $q = k$ and $\mathbf{y} = \mathbf{x}[x_r \mapsto x_r - 1]$ or
  $x_r = 0$, $q = l$ and $\mathbf{y} = \mathbf{x}$.

Notation: $(p, \mathbf{x}) \left|\frac{1}{P}\right. (q, \mathbf{y})$.

Note that if $(p, \mathbf{x})$ is halting (i.e. $I_p = \texttt{halt}$) there is no next configuration. Ditto for $p \geq n$.

Define

$$(p, \mathbf{x}) \left|\frac{0}{P}\right. (q, \mathbf{y}) :\Leftrightarrow (p, \mathbf{x}) = (q, \mathbf{y})$$

$$(p, \mathbf{x}) \left|\frac{t}{P}\right. (q, \mathbf{y}) :\Leftrightarrow \exists\, q', \mathbf{y}' \;\; (p, \mathbf{x}) \left|\frac{t-1}{P}\right. (q', \mathbf{y}') \left|\frac{1}{P}\right. (q, \mathbf{y})$$

$$(p, \mathbf{x}) \left|\frac{}{P}\right. (q, \mathbf{y}) :\Leftrightarrow \exists\, t \;\; (p, \mathbf{x}) \left|\frac{t}{P}\right. (q, \mathbf{y})$$

A computation (or a run) of P is a sequence of configurations C_0, C_1, C_2, ... where $C_i \left|\frac{1}{P}\right. C_{i+1}$.

Note that a computation may well be infinite:

```
0:  inc 0 0
```

has no terminating computations at all. More generally, for some particular input a computation on a machine may be finite, and infinite for other inputs.

Also, computations may get stuck. The program

```
0: inc 0 1
```

cannot execute the first instruction since there is no goto label 1.

Note that we may safely assume that $P = I_0, I_1, \ldots, I_{n-1}$ uses only registers R_i, $i < n$, so all numbers in the instructions are bounded by n.

Furthermore, we may assume that all the goto targets k lie in the range $0 \leq k < n$. Also, I_{n-1} is a halt instruction, and there are no others.

It follows that these clean RMs cannot get stuck, every computation either ends in halting, or is infinite. From now on, we will always assume that our programs are syntactically correct in this sense.

Exercise

Write a program that transforms a RM program into an "equivalent" one that is syntactically correct.

Hence, we have two kinds of computations: finite ones (that necessarily end in a halt instruction), and infinite ones. We will write

$$(C_i)_{i<n} \qquad \text{and} \qquad (C_i)_{i<\omega}$$

for finite versus infinite computations.

ω is the first infinite ordinal, more later. If you don't like ordinals, replace ω by some meaningless but pretty symbol like ∞.

The initial configuration for input $\mathbf{a} \in \mathbb{N}^k$ is $E_{\mathbf{a}} = (0, (\mathbf{a}, \mathbf{0}))$.

Definition

A RMP P computes the partial function $f : \mathbb{N}^k \hookrightarrow \mathbb{N}^\ell$ if for all $\mathbf{a} \in \mathbb{N}^k$ we have:

- If $\mathbf{a}$ is in the domain of f, then the computation of P on $C_0 = E_{\mathbf{a}}$ terminates in configuration $C_t = (n-1, \mathbf{y})$ where $f(\mathbf{a}) = (y_k, \ldots, y_{k+\ell-1})$ and $I_{n-1} = \texttt{halt}$.
- If $\mathbf{a}$ is not in the domain of f, then the computation of P on $E_{\mathbf{a}}$ fails to terminate.

Recall that according to our convention, it is not admissible that an RM program could get stuck (because a goto uses a non-existing label). What if we allowed arbitrary RM programs instead of only clean ones?

The class of computable functions would not change one bit, our definitions are quite robust under (reasonable) modifications. This is a good sign, fragile definitions are usually of little interest.

Exercise

Modify the definition so "getting stuck" is allowed and show that we obtain exactly the same class of partial functions this way. Invent RMs without a halt instruction.

The number of steps in a finite computation provides a measure of complexity, in this case time complexity.

Given a RM P and some input $\mathbf{x}$ let $(C_i)_{i<N}$, where $N \leq \omega$, be the computation of P on $\mathbf{x}$.

We write the time complexity of P as

$$T_P(\mathbf{x}) = \begin{cases} N & \text{if } N < \omega, \\ \omega & \text{otherwise.} \end{cases}$$

If you are worried about ω just read it as ∞. Alternatively, we could use $N - 1$ as our step-count.

This may sound trivial, but it's one of the most important ideas in all of computer science. Period.

To make RMPs slightly easier to read we use names such as X, Y, Z and so forth for the registers.

This is just a bit of syntactic sugar, if you like you can always replace X by R_0, Y by R_1 and so forth.

And we will be quite relaxed about distinguishing register X from its content $[X]$.

There is actually something very important going on here: we are trying to produce notation that works well with the human cognitive system.

Humans are exceedingly bad at dealing with fully formalized systems; in fact, we really cannot read formal mathematics except in the most trivial (and useless) cases. Try reading Russell-Whitehead's Principia Mathematica if you don't believe me.

The current notation system in mathematics evolved over centuries and is very carefully fine-tuned to work for humans.

Computers need an entirely different presentation and it is very difficult to move between the two worlds.

Example: Multiplication

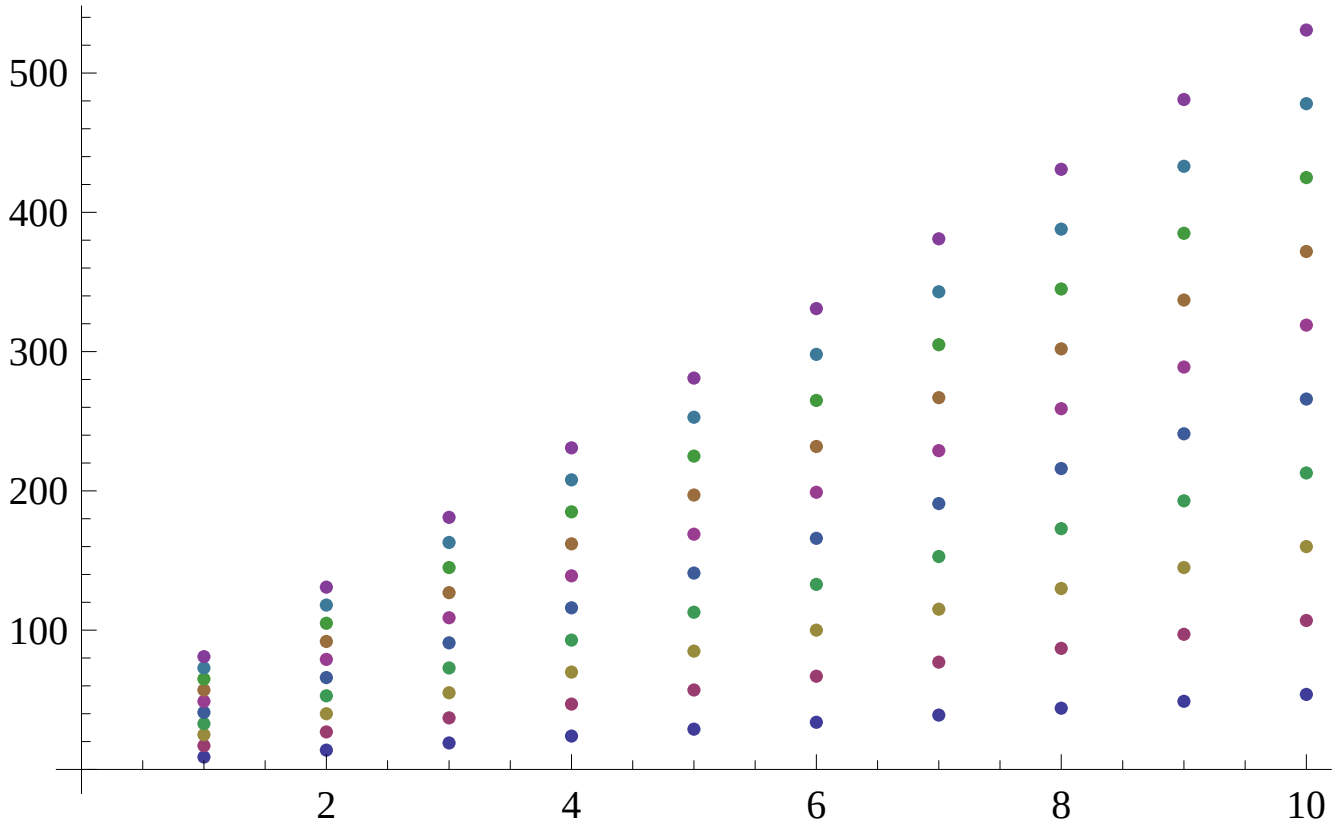
Here is a program that multiplies registers X and Y, and places the product into Z. U is auxiliary.

```
// multiplication   X Y --> Z
  0:   dec X  1  6
  1:   dec Y  2  4
  2:   inc Z  3
  3:   inc U  1
  4:   dec U  5  0
  5:   inc Y  4
  6:   halt
```

A Computation

0	$(2,2,0,0)$	1	$(0,2,2,0)$
1	$(1,2,0,0)$	2	$(0,1,2,0)$
2	$(1,1,0,0)$	3	$(0,1,3,0)$
3	$(1,1,1,0)$	1	$(0,1,3,1)$
1	$(1,1,1,1)$	2	$(0,0,3,1)$
2	$(1,0,1,1)$	3	$(0,0,4,1)$
3	$(1,0,2,1)$	1	$(0,0,4,2)$
1	$(1,0,2,2)$	4	$(0,0,4,2)$
4	$(1,0,2,2)$	5	$(0,0,4,1)$
5	$(1,0,2,1)$	4	$(0,1,4,1)$
4	$(1,1,2,1)$	5	$(0,1,4,0)$
5	$(1,1,2,0)$	4	$(0,2,4,0)$
4	$(1,2,2,0)$	0	$(0,2,4,0)$
0	$(1,2,2,0)$	6	$(0,2,4,0)$

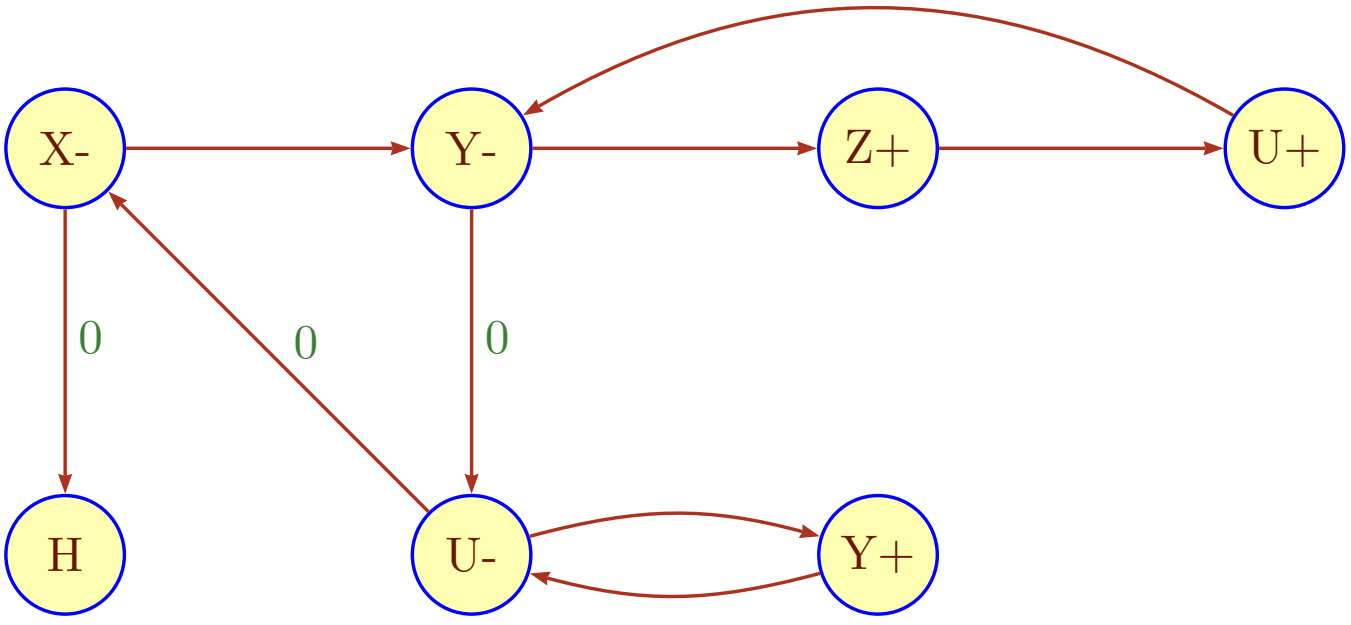
```
// multiplication   X Y --> Z
  0:    dec X  1  6
  1:    dec Y  2  4
  2:    inc Z  3
  3:    inc U  1
  4:    dec U  5  0
  5:    inc Y  4
  6:    halt
```



Exercise

Determine the time complexity of the multiplication RM.

Flowgraph for Multiplication

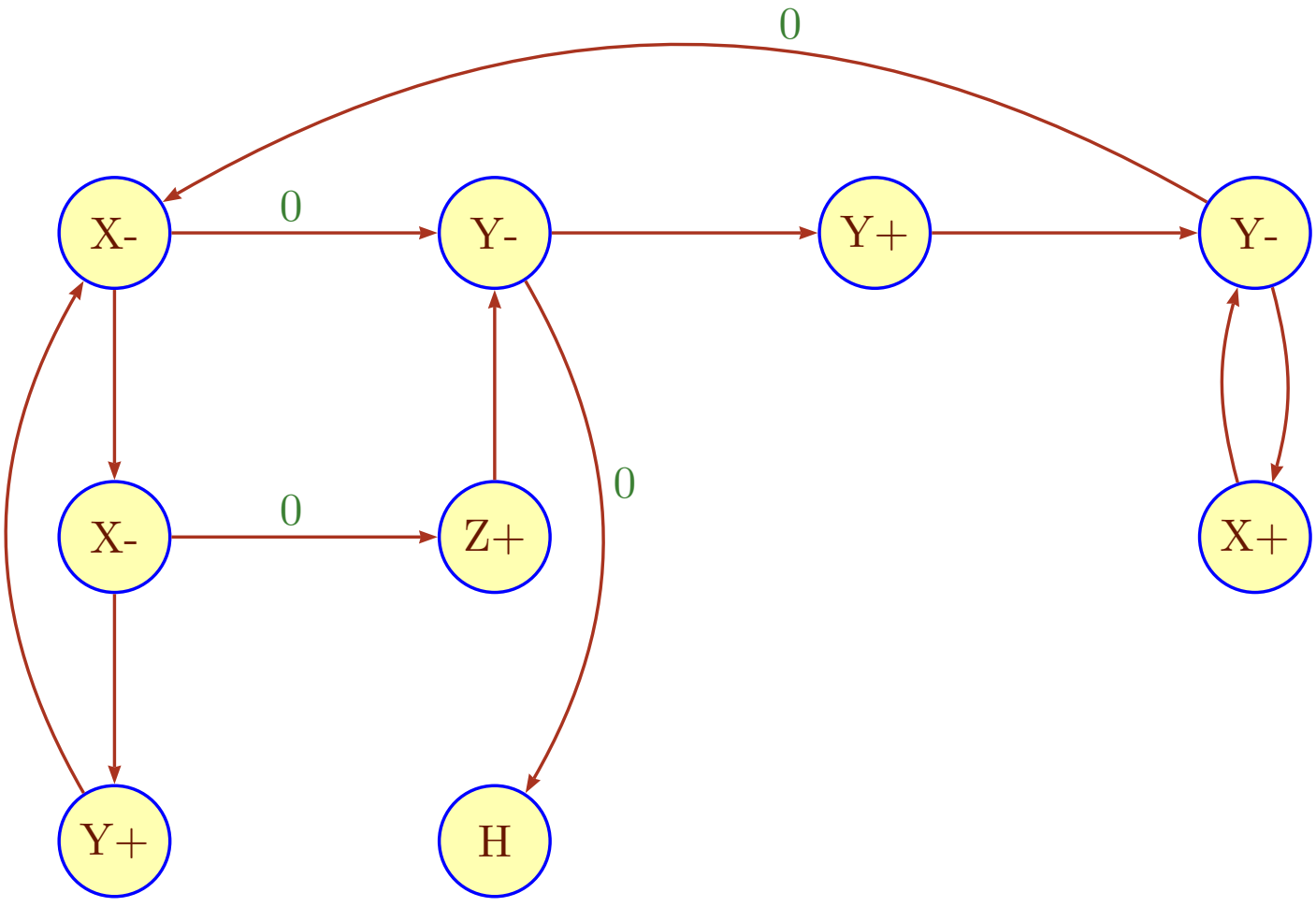


Example: Binary Digit Sum

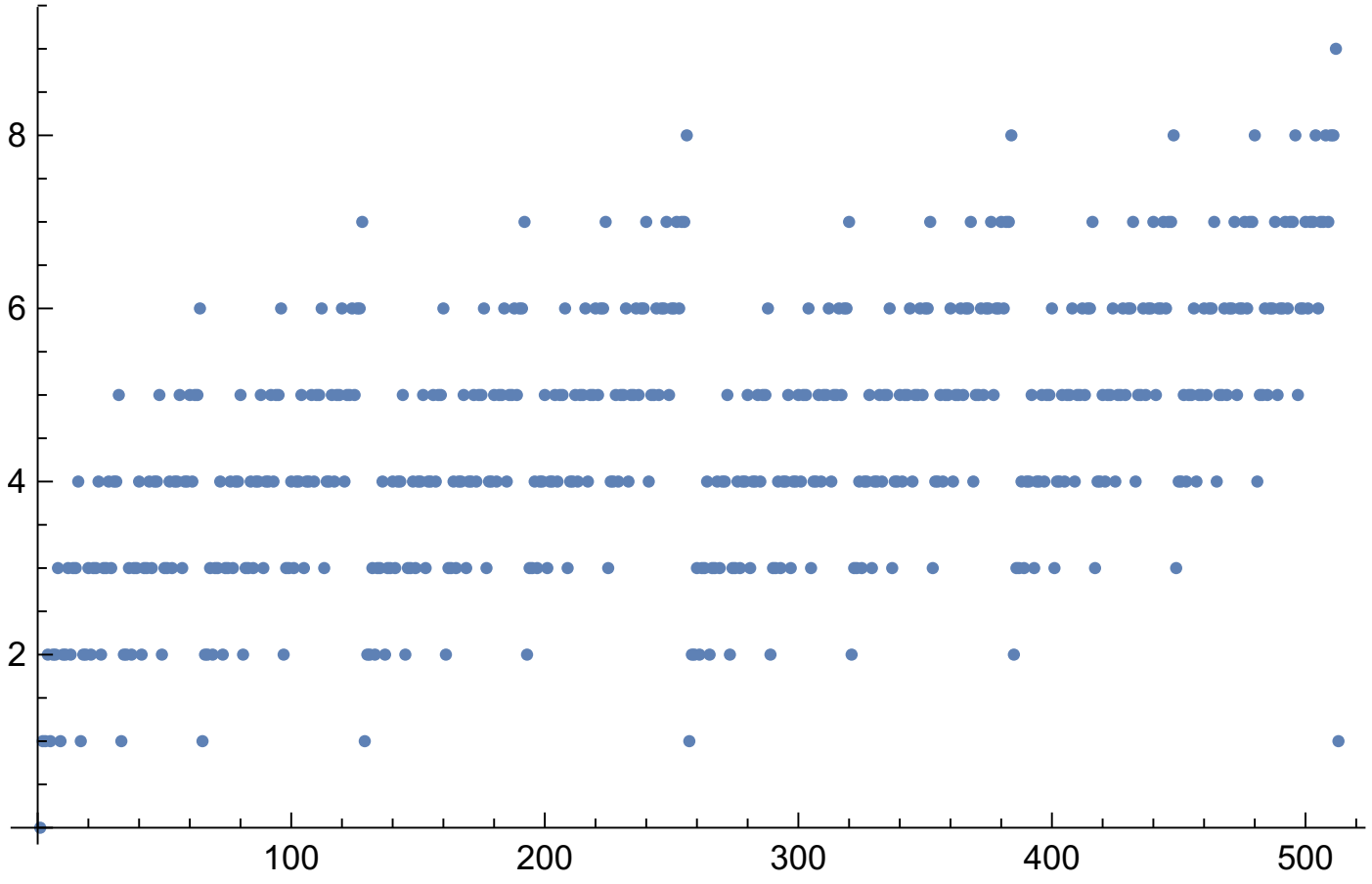
The following RMP computes the number of 1's in the binary expansion of X, the so-called binary digit sum of x.

```
// binary digitsum of X --> Z
  0:    dec X  1  4
  1:    dec X  2  3
  2:    inc Y  0
  3:    inc Z  4
  4:    dec Y  5  8
  5:    inc Y  6
  6:    dec Y  7  0
  7:    inc X  6
  8:    halt
```

Flowgraph for DigitSum



The (binary) digit sum is actually quite useful in some combinatorial arguments.



Exercise

Show that every primitive recursive function can be computed by a register machine. Implement a prec to RM compiler.

Exercise

Suppose some register machine M computes a total function f. Why can we not conclude that f is primitive recursive?

- More Recursion
- Register Machines

3. Coding

- Universality

Since register machines operate only on natural numbers it is not clear how powerful they really are, compared to, say, C programs.

For example, could one concoct a RMP that computes shortest paths in a graph?

We would need to code the graph as a number. Plus all other needed data structures: lists, trees, matrices, hash tables ...

How?

Leopold Kronecker, Semi-Constructivist

Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.

"Dear god" made the integers, everything else is the work of men.

We would like to express a sequence $a_1, a_2, \ldots, a_n$ of natural numbers as a single number $\langle\, a_1, a_2, \ldots, a_n \,\rangle$. So we need a coding function, a polyadic map of the form

$$\langle\, . \,\rangle : \mathbb{N}^* \to \mathbb{N}$$

that is injective, so we can decode: from $b = \langle\, a_1, a_2, \ldots, a_n \,\rangle$ we can recover n as well as all the a_i.

Moreover, both the coding and decoding operations should be computationally cheap: they should be computably by a small RM with small time complexity.

Suppose

$$b = \langle\, a_0, a_1, a_2, \ldots, a_{n-1} \,\rangle$$

is some code number. Note that we have used 0-indexing to simplify notation below.

To organize the decoding process, we want a unary length function $\mathsf{len} : \mathbb{N} \to \mathbb{N}$ that determines the length of the coded sequence

$$\mathsf{len}(b) = n$$

and a binary actual decoding function $\mathsf{dec} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ that extracts the components:

$$\mathsf{dec}(b, i) = a_i$$

for all $i = 0, \ldots, n-1$.

The numbers x of the form $\langle a_0, a_1, a_2, \ldots, a_{n-1} \rangle$ that appear as codes of sequences are called sequence numbers.

Note that a priori $\mathsf{len}(x)$ need not be defined when x is not a sequence number. The same is true for $\mathsf{dec}(x, i)$, plus i may be too large to make sense. Still, one usually insists that both decoding functions are total and return some default value like 0 for meaningless arguments.

Exercise

Show how to check if a number is a sequence number given dec *and* len*.*

The first step is to select a pairing function, an injective map $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Equivalently, we are looking for 3 functions $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $\pi_i : \mathbb{N} \to \mathbb{N}$, $i = 1, 2$, such that

$$\pi_i(\pi(x_1, x_2)) = x_i$$

There are many possibilities, for our purposes the following is arguably the best choice:

$$\pi(x, y) = 2^x(2y + 1)$$

For example

$$\pi(5, 27) = 32 \cdot 55 = 1760 = 11011100000_2$$

Note that the binary expansion of $\pi(x, y)$ looks like so:

$$y_k y_{k-1} \dots y_0 \, 1 \, \underbrace{00 \dots 0}_{x}$$

where $y_k y_{k-1} \dots y_0$ is the standard binary expansion of y (y_k is the most significant digit). Hence the range of π is $\mathbb{N}_+$ (but not $\mathbb{N}$).

This makes it easy to find the corresponding unpairing functions:

$$x = \pi_1(\pi(x, y)) \qquad\qquad y = \pi_2(\pi(x, y)).$$

Aside: Fueter-Pólya

Another popular pairing function is the quadratic polynomial

$$p(x, y) = ((x + y)^2 + 3x + y)/2$$

Note that this function is a bijection (unlike our exponential pairing function which misses 0).

A surprising theorem by Fueter and Pólya from 1923 states that, up to a swap of variables, this is the only quadratic polynomial that defines a bijection $\mathbb{N}^2 \leftrightarrow \mathbb{N}$.

The proof is rather difficult and uses the fact that e^a is transcendental for algebraic $a \neq 0$.

It is an open problem whether there are other bijections for higher degree polynomials. Extra Credit.

$$\begin{aligned} \langle\, \mathsf{nil} \,\rangle &:= 0 \\ \langle\, a_1, \ldots, a_n \,\rangle &:= \pi(a_1, \langle\, a_2, \ldots, a_n \,\rangle) \end{aligned}$$

Here are some sequence numbers for this particular coding function:

$$\begin{aligned} \langle\, 10 \,\rangle &= 1024 \\ \langle\, 0, 0, 0 \,\rangle &= 7 \\ \langle\, 1, 2, 3, 4, 5 \,\rangle &= 532754 \end{aligned}$$

Lemma

$\langle\,.\,\rangle : \mathbb{N}^* \to \mathbb{N}$ *is a bijection.*

Proof. Suppose

$$\langle\, a_1, \ldots, a_n \,\rangle = \langle\, b_1, \ldots, b_m \,\rangle$$

We may safely assume $0 < n \leq m$ (why?).

Since π is a pairing function, we get $a_1 = b_1$ and $\langle\, a_2, \ldots, a_n \,\rangle = \langle\, b_2, \ldots, b_m \,\rangle$.

By induction, $a_i = b_i$ for all $i = 1, \ldots, n$ and $0 = \langle\, \mathsf{nil} \,\rangle = \langle\, b_{n+1}, \ldots, b_m \,\rangle$. Hence $n = m$ and our map is injective.

Exercise

Prove that the function is surjective.

Here is a sequence number and its binary expansion:

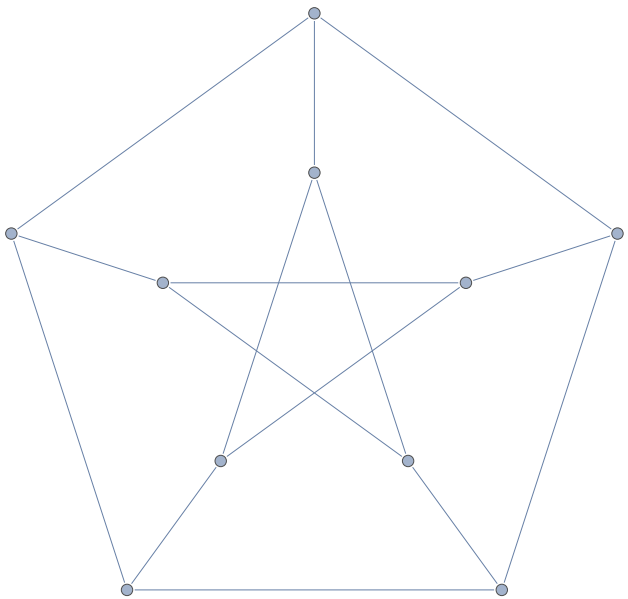
$$\begin{aligned}\langle\, 2,3,5,1 \,\rangle &= 20548 \\ &= 1\underbrace{0}_{1}1\underbrace{00000}_{5}1\underbrace{000}_{3}1\underbrace{00}_{2}\end{aligned}$$

So the number of 1's (the digitsum) is just the length of the sequence, and the spacing between the 1's indicates the actual numerical values.

Our purely arithmetic description is just easier to implement in terms of register machines.

We can now code any discrete structure as an integer by expressing it as a nested list of natural numbers, and then applying the coding function.

For example, the so-called Petersen graph on the left is given by the nested list on the right.



$$((1,3),(1,4),(2,4),(2,5),(3,5),$$
$$(6,7),(7,8),(8,9),(9,10),(6,10),$$
$$(1,6),(2,7),(3,8),(4,9),(5,10))$$

Of course, we also need to be able to operate on sequence numbers.

Exercise

Construct a RMP that computes the length of a sequence, given the code number as input.

Exercise

Construct a RMP that computes the two unpairing functions $x = \pi_1(\pi(x, y))$ *and* $y = \pi_2(\pi(x, y))$

Exercise

Construct a RMP that implements the join of two lists given as sequence numbers.

Exercise

Explain how to compute head and tail functions for sequence numbers.

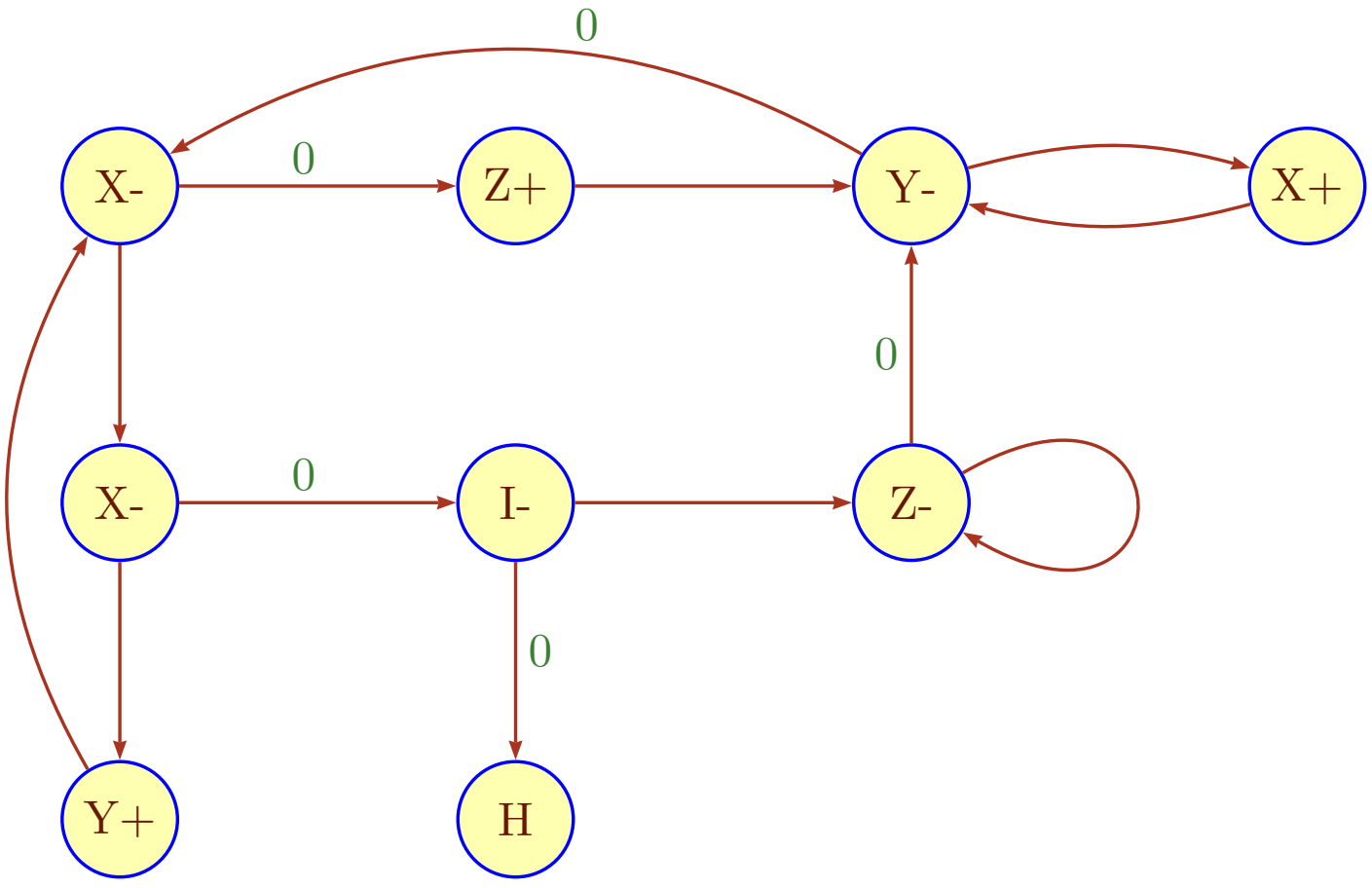
Exercise

Explain how to compute prepend and join operations for sequence numbers.

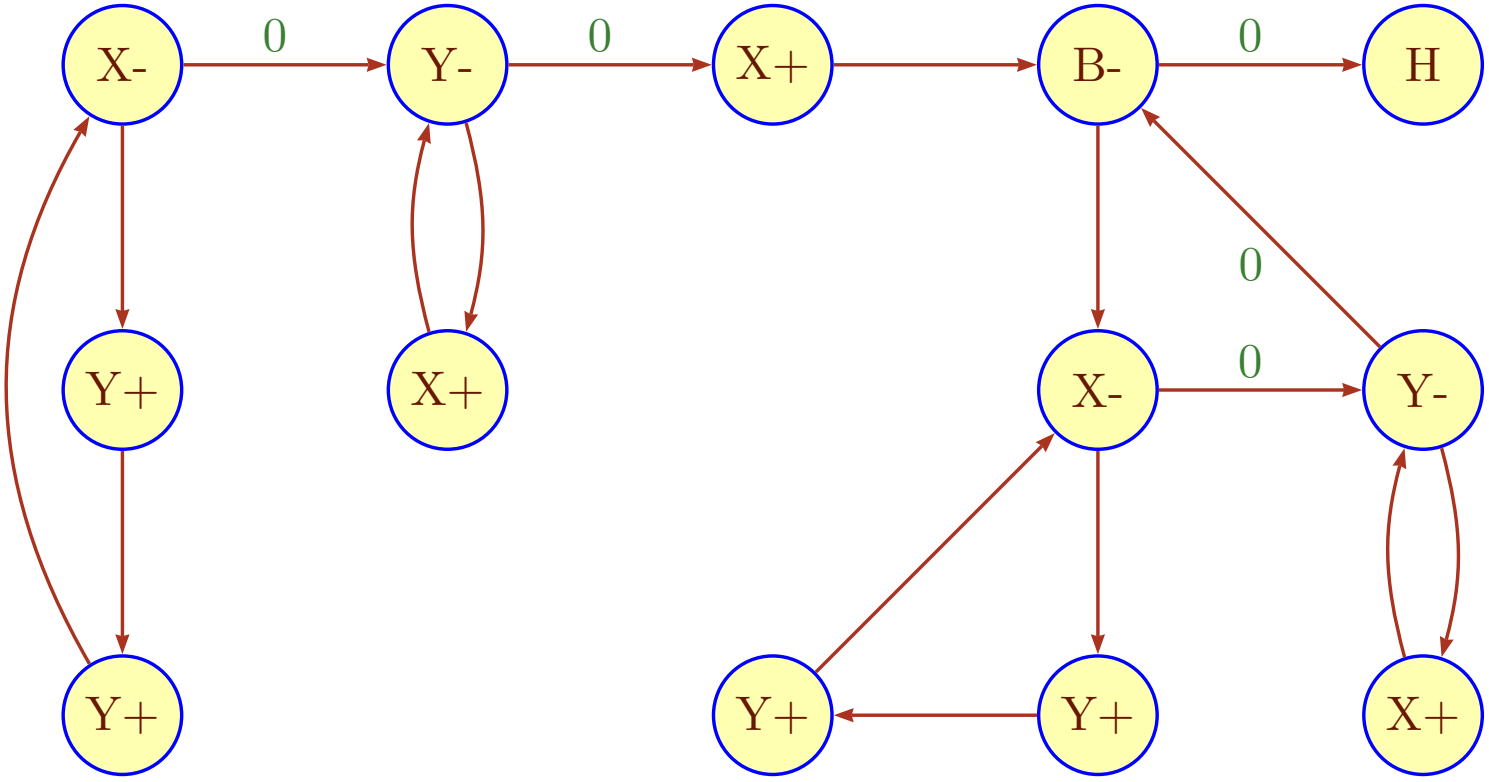
Exercise

Explain how to implement bubble sort using sequence numbers.

Flowgraph dec(x, i)



Prepend b to x



Note that a single instruction of an RMP can also be coded as a number:

- `halt` $\langle 0 \rangle$
- `inc r k` $\langle r, k \rangle$
- `dec r k l` $\langle r, k, l \rangle$

And a whole program can be coded as the sequence number of these numbers.

For example, the simplified addition program

```
// addition   R0 + R1 --> R1
  0:    dec 0  1  2
  1:    inc 1  0
  2:    halt
```

has code number

$$\langle\langle 0,1,2\rangle,\langle 1,0\rangle,\langle 0\rangle\rangle = 88098369175552.$$

Note that this code number does not include I/O conventions, but it is not hard to tack these on if need be.

- More Recursion
- Register Machines
- Coding

4. Universality

As Gödel has shown conclusively, self-reference is a surprisingly powerful tool.

On occasion, it wreaks plain havoc: his famous incompleteness theorem takes a wrecking ball to first-order logic.

However, in the context of computation, self-reference turns into a genuine resource. We developed our coding machinery to show that standard discrete structures can be expressed as natural numbers and thus be used in an RPM. But an RPM is itself a discrete structure, so RPMs can compute with (representations of) RPMs.

This special property of digital computers, that they can mimic any discrete state machine, is described by saying that they are universal machines. The existence of machines with this property has the important consequence that, considerations of speed apart, it is unnecessary to design various machines to do various computing processes. They can all be done with one digital computer, suitably programmed for each case. It will be seen that as a consequence of this all digital computers are in a sense equivalent.

Alan Turing (1950)

Computational universality was established by Turing in 1936 as a purely theoretical concept.

Surprisingly, within just a few years, practical universal computers (at least in principle) were actually built and used:

1941 Konrad Zuse, Z3

1943 Tommy Flowers, Colossus

1944 Howard Aiken, Mark I

1946 Prosper Eckert and John Mauchley, ENIAC

Let's define the state complexity of a RMP to be its length, the number of instructions used in the program.

An RMP of complexity 1 is pretty boring, 2 is slightly better, 3 better yet; a dozen already produces some useful functions. With 1000 states we can do even more, let alone with 1000000, and so on.

Except that the "so on" is plain wrong: there is some magic number N such that every RMP can already by simulated by a RMP of state complexity just N: we can hide the complexity of the computation in one of the inputs. As far as state complexity is concerned, maximum power is already reached at N.

This is counterintuitive, to say the least.

How does one construct a universal computer? According to the last section, we can code a RMP $P = I_0, I_1, \ldots, I_{n-1}$ as an integer e, usually called an index for P in this context.

Moreover, we can access the instructions in the program by performing a bit of arithmetic on the index. Note that we can do this non-destructively by making copies of the original values.

So, if index e and some line number p (for program counter) are stored in registers we can retrieve instruction I_p and place it into register I.

Simulating a RM

Suppose we are given a sequence number e that is an index for some RMP P requiring one input x.

We claim that there is a universal register machine (URM) $\mathcal{U}$ that, on input e and x, simulates program P on x.

Alas, writing out $\mathcal{U}$ as a pure RMP is too messy, we need to use a few "macros" that shorten the program.

Of course, one has to check that all the macros can be removed and replaced by corresponding RMPs, but that is not very hard.

- `copy r s k`
  Non-destructively copy the contents of R_r to R_s, goto k.
- `zero r k l`
  Test if the content of R_r is 0; if so, goto k, otherwise goto l.
- `pop r s k`
  Interpret R_r as a sequence number $a = \langle\, b, c \,\rangle$; place b into R_s and c into R_r, goto k. If $[R_r] = 0$ both registers will be set to 0.
- `read r t s k`
  Interpret R_r as a sequence number and place the $[R_t]$th component into R_s, goto k. Halt if $[R_t]$ is out of bounds.
- `write r t s k`
  Interpret R_r as a sequence number and replace the $[R_t]$th component by $[R_s]$, goto k. Halt if $[R_t]$ is out of bounds.

Here are the registers used in $\mathcal{U}$:

- x input for the simulated program P
- E code number of P
- R register that simulates the registers of P
- I register for instructions of P
- p program counter

Hack: x is also used as an auxiliary variable to keep the whole program small.

```
 0:   copy   E  R  1          // R = E
 1:   write  R  p  x  2       // R[0] = x
 2:   read   E  p  I  3       // I = E[p]
 3:   pop    I  r  4          // r = pop(I)
 4:   zero   I 13  5          // if( I == 0 ) halt
 5:   pop    I  p  6          // p = pop(I)
 6:   read   R  r  x  7       // x = R[r]
 7:   zero   I  8  9          // if( I != 0 ) goto 9
 8:   inc    x 12             // x++; goto 12
 9:   zero   x 10 11          // if( x != 0 ) goto 11
10:   pop    I  p  2          // p = pop(I)
11:   dec    x 12 12          // x--
12:   write  R  r  x  2       // R[r] = x; goto 2
13:   halt
```

Of course, the 13 lines in this universal machine are a bit fraudulent, we really should expand all the macros. Still, the resulting honest register machine would not be terribly large.

And there are lots of ways to optimize.

Exercise

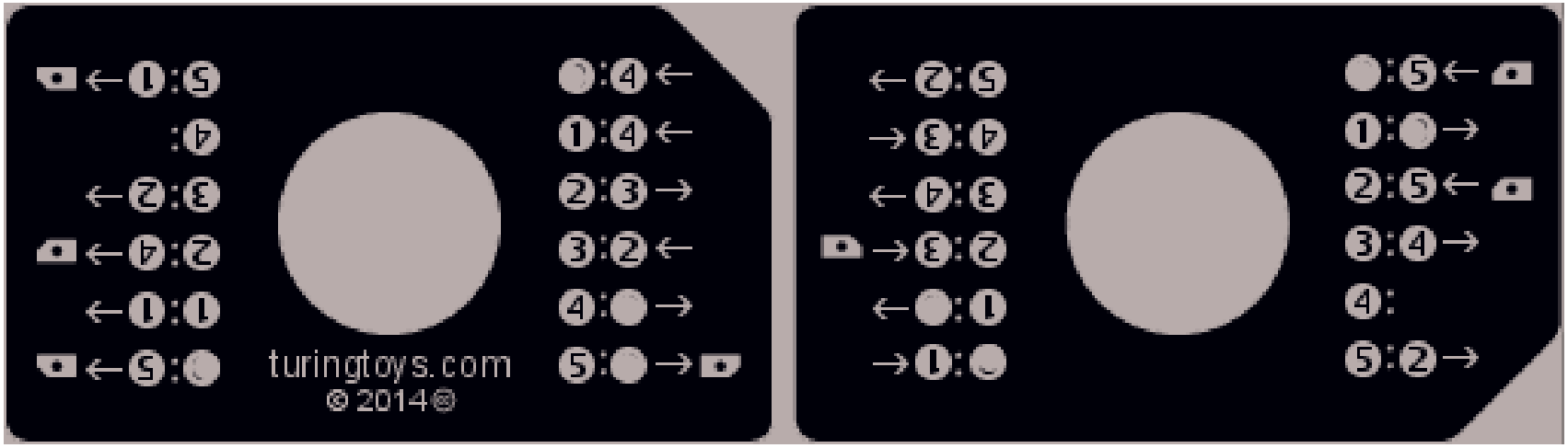
Give a reasonable bound for the size of the register machine obtained by expanding all macros.

Exercise

Try to build a smaller universal register machine.

If we define computability in terms of RMs, it follows that the Halting Problem for RMs is undecidable: there is no RM that takes an index e as input and determines whether the corresponding RM P_e halts (on all-zero registers).

Since RMs are perfectly general computational devices, this means that there is no algorithm to determine whether RM P_e halts; the Halting Problem is undecidable.



Exercise

Figure out what this picture means.

Exercise (Very Hard)

Prove that this is really a universal Turing machine.