Constructive Logic (15-317), Fall 2014
Assignment 9: Modal Logic and Lax Logic

Joe Tassarotti (`jtassaro@andrew`), Evan Cavallo (`ecavallo@andrew`)

Out: Thursday, November 13, 2014
Due: Thursday, November 27, 2014 (before noon)

In this assignment, you will explore modal logic, which adds the necessity and possibility modalities as new connectives, and lax logic, which adds the lax modality.

Your work should be submitted electronically before the beginning of the class. Please convert your homework to a PDF file titled `hw09.pdf`, and put the file in

```
/afs/andrew/course/15/317/submit/<your andrew id>
```

If you are familiar with LaTeX, you are encouraged to use this document as a template for typesetting your solutions, but you may alternatively write your solutions *neatly* by hand and scan them.

1 A Few Truths (15 points)

Task 1 (6 pts)**.** Give derivations of the following judgments in modal logic:

1. $(\Box A \vee \Box B) \supset \Box(A \vee B)$ *true*
2. $\Diamond(A \wedge B) \supset \Diamond A \wedge \Diamond B$ *true*
3. $\Box(A \supset B) \supset \Diamond A \supset \Diamond B$ *true*

Task 2 (9 pts)**.** Give derivations of the following judgments in lax logic:

1. $\bigcirc(A \Rightarrow B) \Rightarrow \bigcirc A \Rightarrow \bigcirc B$ *true*
2. $\bigcirc \bigcirc A \Rightarrow \bigcirc A$ *true*
3. $(A \vee \neg A) \Rightarrow \neg \bigcirc \bot \Rightarrow (\bigcirc A \Rightarrow A)$ *true*

2 Lax Logic in Modal Logic (16 points)

The introduction and elimination rules for the lax modality are very similar to the rules for possibility. The difference lies in the second premise for the elimination rule.

$$\frac{\Delta; \Gamma \vdash \diamond A\ \mathit{true} \quad \Delta; A\ \mathit{true} \vdash C\ \mathit{poss}}{\Delta; \Gamma \vdash C\ \mathit{poss}}\ \diamond E \qquad \frac{\Gamma \vdash \bigcirc A\ \mathit{true} \quad \Gamma, A\ \mathit{true} \vdash C\ \mathit{lax}}{\Gamma \vdash C\ \mathit{lax}}\ \bigcirc E$$

For the lax modality, we can use our truth assumptions Γ along with A *true* to prove C *lax*; with possibility, we only have our valid assumptions. This suggests a translation of lax logic into modal logic, where (in particular) assumptions in lax logic become validity assumptions in modal logic. We define a translation $()^+$ going from lax to modal logic as follows.

On propositions:

$$\begin{array}{rcll}
(\bigcirc A)^+ & = & \diamond\Box A^+ & \\
(A \Rightarrow B)^+ & = & \Box A^+ \supset B^+ & \\
(A \wedge B)^+ & = & A^+ \wedge B^+ & \\
(A \vee B)^+ & = & A^+ \vee B^+ & \\
\top^+ & = & \top & \\
\bot^+ & = & \bot & \\
P^+ & = & P & \text{for atomic } P
\end{array}$$

On contexts:

$$\begin{array}{rcl}
(\cdot)^+ & = & \cdot \\
(\Gamma, A\ \mathit{true})^+ & = & \Gamma^+, A^+\ \mathit{valid}
\end{array}$$

Theorem 1 (Lax Logic in Modal Logic)**.** *Let $\vdash^L$ denote lax entailment and $\vdash^M$ denote modal entailment. Then*

1. *If $\Gamma \vdash^L A$ true then $\Gamma^+; \cdot \vdash^M A^+$ true,*
2. *If $\Gamma \vdash^L A$ lax then $\Gamma^+; \cdot \vdash^M \Box A^+$ poss.*

Task 3 (16 pts)**.** Prove Theorem 1 by induction on the derivation in lax logic. You need only show the cases for $\bigcirc I$, $\bigcirc E$, $\Rightarrow I$, and $\Rightarrow E$.

3 One Interpretation (12 points)

Double-negation is one example of a monad, the type of operation captured by $\bigcirc$ in lax logic. We can therefore interpret lax logic in standard constructive logic by translating the lax modality $\bigcirc$ as double-negation:

On propositions:

$$
\begin{aligned}
(\bigcirc A)^* &= \neg\neg A^* \\
(A \Rightarrow B)^* &= A^* \supset B^* \\
(A \wedge B)^* &= A^* \wedge B^* \\
(A \vee B)^* &= A^* \vee B^* \\
\top^* &= \top \\
\bot^* &= \bot \\
P^* &= P \qquad \text{for atomic } P
\end{aligned}
$$

On contexts:

$$
\begin{aligned}
(\cdot)^* &= \cdot \\
(\Gamma, A\ \mathit{true})^* &= \Gamma^*, A^*\ \mathit{true}
\end{aligned}
$$

Theorem 2 (Lax Modality as Double Negation)**.**

1. *If* $\Gamma \vdash^L A$ *true then* $\Gamma^* \vdash A^*$ *true,*
2. *If* $\Gamma \vdash^L A$ *lax then* $\Gamma^* \vdash \neg\neg A^*$ *true.*

Task 4 (8 pts)**.** Prove Theorem 2 by induction on the derivation in lax logic. You need only show the cases for $\bigcirc I$ and $\bigcirc E$.

We have the following easy corollary:

Corollary 1. Let A be a proposition of lax logic in which the lax modality $\bigcirc$ does not appear. If $\cdot \vdash_L \bigcirc A$ *true*, then $\cdot \vdash_L \neg\neg A$ *true*.

Task 5 (4 pts)**.** Is the converse true? In other words, given A as above and $\cdot \vdash_L \neg\neg A$ *true*, will $\cdot \vdash_L \bigcirc A$ *true* necessarily hold? If not, give a counterexample (and explain informally why it is a counterexample).